Collective Effects on the Growth of Vapor Bubbles in a Superheated Liquid

G. L. Chahine
H. L. Liu
Tracor Hydronautics, Inc., Laurel, MD

Introduction

In hydrodynamic applications as with propellers, pumps, channels, pipes, submerged jets, etc., cavitation occurs when the liquid experiences significant pressure drops due to local high flow velocities. In this case the growth of the nuclei to macroscopic bubbles is mainly controlled by inertia. Heat transfer during the phase change at the bubble wall is negligible because of the low value of the vapor density at the ambient temperatures, and does not affect the bubble-wall motion. The phase change is then called "cavitation" and the liquid is described as "cold." Conversely in applications where heat exchange, rather than the liquid inertia, controls the vapor-bubble dynamics, the phase change is called "boiling."

In many modern processes such as power plants and nuclear engineering the liquids involved are in such conditions that the liquid is described as "superheated" and both heat transfer as well as inertia contribute to controlling the bubble behavior. Phase change and cavity growth appear in these liquids in applications such as high speed flow of sodium cooled fast-breeder reactors in nuclear power engineering, and flow of hot water in nozzles and pipes in steam power plants. Accidents such as loss of coolant, or rupture of a pipe, are sources of boiling nucleation and evidently of major safety concern.

The problem of the growth of a spherical isolated bubble in a superheated liquid has been extensively studied. However, very little work has been done for the case of a cloud of bubbles. The collective behavior of the bubbles departs considerably from that of a single isolated bubble due to the cumulative modification of the pressure field from all other bubbles. This paper presents a theoretical study on bubble interaction in a superheated liquid during the growth stage. The solution is sought in terms of matched asymptotic expansions in powers of $\epsilon$, the ratio between $r_0$, a characteristic bubble radius and $l_0$, the interbubble distance. Numerical results show a significant inhibition of the bubble growth rate due to the presence of interacting bubbles. In addition, the temperature at the bubble wall decreases at a slower rate. Consequently, the overall heat exchange during the bubble growth is reduced.

Formulation of the Problem

Let us consider the case of low void fraction in an unbounded medium of uniform pressure, $P_\infty$ and temperature $T_\infty$. We assume the fluid to be inviscid and incompressible and the flow to be irrotational. These assumptions are usual and can be justified in boiling heat transfer and cavitation studies except in the last phase of the bubble collapse and in the very early phase of bubble growth where the corresponding Reynolds number is small. At $t = 0$ a sudden variation in the ambient pressure, $P_\infty(t)$, is introduced and the cloud dynamics is sought. Provided that the characteristic size $r_0$ of a bubble is small compared to its characteristic distance $l_0$ from its neighbors, we can assume that interbubble interactions are weak enough so that, in the first order of approximation, and in the absence of relative velocity with the surrounding fluid, each of the individual bubbles reacts to the local pressure variations spherically, as if isolated. Mutual bubble interactions, individual bubble or cluster has been studied recently and papers on this subject were presented at the ASME meetings the past two years [3, 4]. We proposed to the National Science Foundation to use the same approach we have developed to study the inertial behavior of a multibubble system to investigate bubble interaction in a superheated liquid. We present here the first results of this study. The collective behavior of the bubble is expected to depart significantly from that of a single isolated bubble due to the cumulative modification of the pressure field at the location of a bubble due to the other bubbles [4, 5]. However, this effect which is very important during the collapse phase is expected to be more moderate during the growth phase. The bubble departure from sphericity is also expected to be smaller in the growth case.
The solution of the problem is sought in terms of matched asymptotic expansions in powers of $\epsilon$, the ratio between $r_{b_0}$ and $l_0$. Two regions of the fluid are defined for each individual bubble of the cloud. In the "inner region," of scale $r_{b_0}$, boundary conditions on pressures, velocities and heat transfer can be written at the bubble wall. In the "outer region," of scale $l_0$, the cloud appears as a distribution of singularities, whose strength is determined by solving the "inner problem." In each order of approximation an "inner problem" reduces to the study of an isolated bubble with conditions imposed at infinity and determined from the "outer problem" at the preceding order. By application of the matching principle these conditions are obtained as the expansions of the preceding order "outer solution" near the bubble singularity. The process is started by the first order approximation whose solution is known since all bubbles behave then as if isolated. Knowing the behavior to this first order of all the bubbles, one obtains the solutions of the first order "outer problem." The corresponding potential is the sum of the potentials due to the sources representing all the spherical bubbles. These sources are each located at a bubble center and their intensity is related to the bubble radius and the growth rate. The behavior of this first order "outer solution" in the vicinity of a particular bubble provides, through the matching condition, the condition at infinity to satisfy when solving the second order "inner problem" corresponding to this bubble. Obtaining the solution for all second order "inner problems" one can pursue the same procedure to solve all following higher orders. This approach is similar to the one we have developed for the study of the collapse of two bubbles [8] and that we have extended to a bubble cloud [7]. These two earlier papers [7, 8] give more details about the method concerning the expansions at each order and the matching procedure.

With our above assumptions both inner and outer flows are potential flows. Practically this applies for the usual ranges of Reynolds numbers where the liquid can be considered as inviscid. The equations of the problem are the following. In the inner problem in order to determine the flow field around any individual bubble, as well as its motion and deformation in the cloud, one has to solve the Laplace equation

$$\Delta \Phi_{in} = 0,$$

subjected to kinematic and dynamical conditions on the bubble's surface.

$$\nabla \Phi_{in} \cdot n' = \frac{1}{\rho_{b_0} c'_{in}} [\hat{r} e_\hat{r} + b e_\theta] n'$$

$$\rho \left[ \Phi_{in} - \hat{b} e_\hat{r} + 1/21 \nabla \Phi_{in} \cdot n \right] = P_m(t) - P_{in} - P_{c'_{in}} + 2\gamma c'(t, \theta, \phi)$$

where $C'$ and $n'$ are, respectively, the local curvature of the surface of bubble $B'$ and its unit vector at the point $M(r, \theta, \phi)$. The equation of the bubble surface in a coordinate system moving with velocity $b'$ in the direction $e_\hat{r}$, is $r = R(\theta, \phi, t)$. $\gamma$ is the surface tension and dots denote time differentiation. $\Phi_{in}$ and the operator $\nabla$ are expressed in the moving coordinates system. $P_c$ can be related to its initial value, $P_{c_{in}}$, and to the bubble volume $V$ by the equation

$$P_{c}(t) = P_{c_{in}} \left[ \frac{\rho(0)}{\rho(t)} \right]^{2/3},$$

where $V = \nabla \Phi_{in}$ is the fluid velocity given by equations (1-4), and $D$ is the thermal diffusivity of the liquid. Equation (5) is subjected to a boundary condition at the bubble wall stating that the heat locally lost at any point of the interface is used to vaporize an amount of liquid determined by the local bubble volume expansion rate. This can be written:

$$\frac{\partial T}{\partial n'} \bigg|_{r=R(t), \phi_{in}} = \frac{\sigma_{in} L}{K} R_i^4$$

In the outer problem, one needs to be concerned with the macrobehavior of the cloud. Here the bubbles appear only as singularities (dynamic sources and heat sinks) whose intensities are determined by the resolution of the various inner problems. The equations of the problem are therefore limited in the outer region to the Laplace equation

$$\Delta \Phi_{out} = 0,$$

only submitted to the at-infinity (far from the cloud) boundary conditions in addition to the presence of the distributed singularities condition between the inner and the outer problem can be simply stated:

$$\Phi_{out} = \text{behavior at } \phi_{in}$$

In order to make asymptotic expansions (and thus to compare orders of magnitudes) an accurate choice of characteristic scale variables is fundamental. For the length scales the choice is immediate: $r_{b_0}$ in the inner problem, $l_0$ in the outer. However, the relationship between $r_{b_0}$ and the characteristic initial bubble radius, $R_0$, is less obvious. Indeed, while in the case of bubble collapse, the bubble radius stays of order $R_0$ in the mathematical sense, $|R(\epsilon) - \epsilon R_0|$; if there exists a constant $\lambda independent of $\epsilon$ such that $|R(\epsilon) - \lambda R_0|$, this is not the case for the bubble cloud growth studied here. The problem is approached in the following way. $r_{b_0}$ is chosen arbitrarily (independent of $\epsilon$) such that the inequality, $r_{b_0}/l_0 = \epsilon << 1$ is valid. When doing this we must keep in mind that the results of the computations will be valid only as long as the radius of any bubble in the cloud does not exceed too much $r_{b_0}$. Concerning the time scale, the choice is simple once $r_{b_0}$ is known. For the problem of a sudden depressurization, this time scale is related to the pressure drop, $\Delta P$, by the relation: $\tau_0 = r_{b_0} (\rho/\Delta P)^{n}$. In the numerical examples considered here the pressure drops from its initial value, $P_{in}$, to a constant value $P_{out}$, in which case $\Delta P$ is defined as: $\Delta P = P_{in} - P_{out}$. The equation of the surface of any bubble in the cloud is expanded in spherical harmonics and can be shown [4, 5] to be written up to the order $\epsilon^3$ as follows

$$\vec{R}(\theta, t) = a_0(t) + a_1(t) + \epsilon^2 \left[ a_2(t) + f_2(t) \cos \theta \right] + \epsilon^3 \left[ a_3(t) + f_3(t) \cos \theta + g_3(t) \phi(\cos \theta) \right]$$

where $\phi_2$ is the Legendre polynomial of order 2 and $\theta$ is the angle seen from the bubble center between the direction of a field point on the bubble surface and the "center of the cloud" (direction of motion of the bubble) [5, 7]. The nondimensional radius $a_0(t)$ (reference length, $r_{b_0}$) is given by the Rayleigh-Plesset equation:

$$a_0 \frac{d a_0}{d t} + \frac{3}{2} \frac{d^2 a_0}{d \theta^2} = -\ddot{P}_{in}(t) + \tau_0(t) +$$

$$-\sigma - \frac{2W_{a_0}^{-1}}{a_0} + \left( \sigma + \frac{2W_{a_0}^{-1}}{R_0} \right) \left( \frac{a_0}{R_0} \right)^{-1}$$

where $R_0$ is the initial nondimensional bubble radius and $k$ the polytropic gas coefficient. The nondimensional parameters are given by:

$$\text{DECEMBER 1984, Vol. 106 / 487}$$
where $P_{\text{a}}(t)$ is the imposed ambient pressure and $\Delta P$ is the characteristic size of the pressure variations. $\gamma(t)$ and $\rho_{\text{a}}(t)$ are the surface tension coefficient and the vapor pressure at the temperature of the bubble wall at time $t$. When the temperatures at the surface of the bubble depart significantly from the ambient, and for liquids where the dependence of $\rho_0$ and $\gamma$ on temperature is important, it is necessary to couple equation (9) with the heat equation to obtain a solution. As an example, for a 1 mm bubble in water at 20°C the temperature drop, $\Delta T$, 1 ms after the beginning of the growth process is 0.2 degrees and $P_0$ drops 1 percent. However, near 100°C, $\Delta T$ is 13 degrees and $P$ decreases by half [6]. In the first case water is "cold" and there is no need to solve for the temperature. This is, however, necessary for "hot" water. The dependence of $\gamma$ on temperature is very weak and could therefore be neglected. The Weber number, $W_e$, will therefore be considered constant in the following.

Once equations (9), (11), and (12) are solved the first approximation of the variation with time of any bubble radius, $a_0^{(i)}$, in the cloud is determined. This allows us to proceed as described in the preceding paragraph, to determine at the following order the boundary conditions at infinity for the inner problem concerning any bubble, $B^{(i)}$. This is obtained by expanding the outer solution obtained at order zero in the vicinity of $B^{(0)}$. To the first order, $\epsilon$, the correction amounts to just a change of the ambient pressure for the inertia problem. This implies a correction of the strength of both the dynamical source and the thermal sink representing $B^{(0)}$ in the outer problem. The "inner problems" remain, therefore, spherical to this order and in absence of condensables the bubble radius correction, $a_1^{(i)}$, is given by the following equation where the subscript $(i)$ has been omitted for convenience.

$$a_0 a_1 + 3a_0 a_2 + a_1 \left( \frac{2 W_e^{-1}}{a_0^2} \right)$$

$$= - \sum_j \left( \frac{\partial}{\partial \eta_j} \right)^2 \phi_j^{(i)} + \pi_1(t),$$

where $\eta_j$ is the distance between the bubbles $B^{(j)}$ and $B^{(0)}$, and $\pi_0^{(j)}(t)$ is the strength of the oscillating source in $B^{(j)}$. $\pi_1(t)$ is a correction of $\pi_0(t)$ and expresses the second approximation of the vapor pressure at the bubble wall. With the temperature expanded in powers of $\epsilon$ as $R(\theta, t)$,

$$T(r, \theta, t) = T_0(r, t) + \epsilon T_1(r, t) + \epsilon^2 \left[ T_2(r, t) + T_3(r, t) \cos \theta \right] + \epsilon^3 \left[ T_4(r, t) + T_5(r, t) \cos \theta + T_6(r, t) \beta_2(\cos \theta) \right] + o(\epsilon^4),$$

$$\pi_0(t)$$

and $\pi_1(t)$ can be expressed as:

$$\pi_0(t) = \left[ \rho_0 \left( \rho_0(a_0, t) - \rho_0(T_0) / \Delta P \right) \right],$$

$$\pi_1(t) = \left[ \rho_0 \left( \rho_1(a_0, t) + a_1 \frac{\partial T_0}{\partial t} (a_0, t) \right) \right] / \Delta P,$$

Since the "inner problem" is spherical at this order and at the preceding one, the heat equation can be written after using the Lagrange transformation:

$$y = \frac{1}{3} \left[ r^2 - R^3(t) \right],$$

$$\frac{\partial T_0}{\partial t} = P_e^{-1} \frac{\partial}{\partial y} \left( \frac{\partial T_0}{\partial y} \right); \quad n = 0, 1,$$

The heat balance at the bubble wall can be expressed with the same variables as follows:

$$a_0 \frac{\partial T_0}{\partial y} \bigg|_{y=0} = \alpha a_0,$$

$$a_0 \left( \frac{\partial T_1}{\partial y} + 2 \frac{a_1}{a_0} \frac{\partial T_0}{\partial y} \right) \bigg|_{y=0} = \alpha a_1.$$

Following the same procedure, once $a_0^{(i)}$ and $a_1^{(i)}$ are known, one can determine the governing equations of motion at higher orders of approximation. For instance, the order $\epsilon^2$ correction will reflect first, a change of the strength of the source located in $B^{(j)}$ due to the source strength corrections at the preceding order of the other singularities located in $B^{(i)}$. Secondly a nonspherical correction is to be applied which is due to an imposed uniform velocity which results from compounding the velocities induced by all the other bubbles $B^{(j)}$ at the location of $B^{(i)}$. For this reason, the nonspherical component of the velocity potential which is in cos $\theta$, can be interpreted as both a translation velocity and a deformation of the bubble. At order $\epsilon^3$, another additional nonspherical correction appears, which is due to a velocity gradient generated by the flow field associated with the motion of the other bubbles.
The dynamical equations at order $\varepsilon^2$ can be written:

$$a_0 \ddot{a} + 3a_0 \dot{a} + a_2 \left( \ddot{a} - \frac{2W}{a_0^2} \right)$$

$$+ \left( \frac{3}{2} \frac{a_0^2}{a_1} + a_1 \dot{a} + 2W \frac{1}{a_0^2} \right)$$

$$= \tau_2(t) - \sum_i \left( \frac{L_i}{R_i^2} \right) q_i^0,$$  \hspace{1cm} (24)

$$a_0 \ddot{f} + 3a_0 \dot{f} = \tau_2(t) +$$

$$-3 \sum_i \left( \frac{L_i}{R_i^2} \right)^2 \left( \dot{a}_0 q_i^{(0)} + a_0 \dot{q}_i^{(0)} \right),$$  \hspace{1cm} (25)

where $q_i^{(0)}$ and $q_i^{(1)}$ are the strengths of the sources of order 1 in $\varepsilon$ (related to $a_i^{(0)}$ and $a_i^{(1)}$). The energy equation is then expanded as follows:

$$\dot{T}_1 = P_e^{-1} \frac{\partial}{\partial y} \left[ 4a_1 a_2 \eta^{1/3} \frac{\partial T_0}{\partial y} + \eta^{4/3} \frac{\partial T_1}{\partial y} \right],$$  \hspace{1cm} (26)

$$\dot{T}_2 + \frac{\partial T_0}{\partial y} [q_2 - a_0 a_1 - a_0^2 a_1 - a_0^3]$$

$$= P_e^{-1} \left\{ \frac{\partial}{\partial y} \left[ \eta^{4/3} \frac{\partial T_2}{\partial y} \right] \right\},$$

$$+ \frac{\partial}{\partial y} \left[ (4a_1 a_2 + 4a_2^2 a_0) \eta^{1/3} + 2a_1 a_2^2 \eta^{2/3} \right] \frac{\partial T_0}{\partial y},$$

$$+ \frac{\partial}{\partial y} \left[ 4a_1 a_2^2 \eta^{1/3} \frac{\partial T_0}{\partial y} \right],$$

$$\dot{T}_3 + \frac{\partial T_0}{\partial y} \left[ -2a_2 \eta^{1/3} - \sum_i \left( \frac{L_i}{R_i^2} \right)^2 q_i^{(0)} \eta^{2/3} \right]$$

$$= P_e^{-1} \left[ \eta^{-2/3} \left( -2T_2 + 2f_2 a_0 \frac{\partial T_0}{\partial y} \right) \right]$$

$$+ \frac{\partial}{\partial y} \left( 4f_2 a_0 \eta^{1/3} \frac{\partial T_0}{\partial y} \right),$$  \hspace{1cm} (27)

where, $\eta = \frac{a_0^2}{3y}$, and $q_i$ and $h_i$ are terms appearing in the expressions of the potential flow $\Phi_i$. The heat balance on the bubble interface can be written:

$$a_0^2 \frac{\partial T_2}{\partial y} + 2a_0 \frac{\partial T_0}{\partial y} + \left( \frac{a_0}{a_0^2} + \frac{a_1}{a_0^2} \right) \frac{\partial T_0}{\partial y} \bigg |_{y=0} = \Theta \Delta \gamma_2$$

$$2f_2 \frac{\partial T_0}{\partial y} \bigg |_{y=0} = \Theta \Delta \gamma_2,$$  \hspace{1cm} (28)

As for the dynamical problem [4, 7] we have derived the expansions up to and including the order $\varepsilon^3$. In order to be brief we will not present here any further expansions.

**Approximate Solution: Thermal Boundary Layer**

In order to solve the various equations of the problem one needs to couple the resolution of the dynamical problem (by a multi-Runge-Kutta procedure for example) to a resolution method for the heat equation. We are presently considering two approaches to solve the equations appearing at each order of approximation. The first approach, similar to that used in [2] for an isolated bubble, is exact in that it does not consider any assumption on the thickness of the thermal boundary layer at the bubble wall. It consists of solving the heat equation in the fluid through a space marching procedure. The domain of applicability of this method is very large but the corresponding computing time could be considerable.

The second method is based on the assumption that the amount of superheat is large enough so that the distance $\delta$ in which the temperature rises from the bubble wall temperature to the ambient temperature is small relative to the bubble radius. In that case a thermal boundary layer assumption is introduced [1] and its thickness can be approximated by:
\[
\delta = \frac{\rho_v}{\rho} \left( \frac{L}{C(T_w - T_b)} \right) = \frac{1}{J'}
\]  

(31)

where \( C \) is the specific heat of the liquid and \( \rho \) its density (the other variables have been defined earlier). An analytical expression of the temperature at the bubble wall has been derived in this case and can be written for a sphere of radius \( a(t) \),

\[
T_w = T_{s0} - \sqrt{\frac{2}{K} \int_0^t \frac{L\rho_v}{a(\tau)^3} \left( \int_0^\tau a(\eta) d\eta \right)^{\frac{1}{2}} d\tau}
\]  

(32)

Prosperetti and Plesset [1] showed that this expression is valid for Jacob numbers, \( J \), as low as 3. This same equation applies for \( T_w(a, \theta) \) and \( a_0(t) \) in our approach and similar equations are derived at the following orders [5]. We will not consider here this relatively accurate but laborious approach. We will rather consider a further simplification for the study of collective bubble growth. Solutions of the dynamical equation in the absence of heat transfer show that during the growth phase bubble deformations are negligible at least as far as the time history is as the asymptotic method is valid. Figure 1 presents the example of the growth of initially spherical bubbles of initial nondimensional radius \( 10^{-4} \), symmetrically located on a sphere. For the case of an isolated bubble, the growth follows the asymptotic linear behavior. However, when the number of interacting bubble increases, the pressure field associated with the growth of the other bubbles in the cloud reduces the growth rate. Finally for \( N = 12 \), the method apparently fails for \( t > 0.1 \), since the radius corrections become larger than the order zero radius. Figure 2 shows, for the same bubble configuration, the ratio of the deformations of type \( \cos \theta \) to the spherical part of \( R(\theta, t) \) in the expansion (1). In all cases but the obvious case where the method breaks down the deformations are less than 4 percent where the bubble has attained 2,000 times its initial radius. This result depends obviously, in general, on \( R_0 \) and \( l_0 \) as well as on the number of interacting bubbles, \( N \). The deformations are larger for larger \( R_0/l_0 \) and \( N \). Therefore, based on the above observation we can consider equation (32) to be applicable for the corrected bubble radius, \( a_\epsilon(t) \) for small \( \epsilon \), and as long as deviations from sphericity are negligible.

With this simplification, at any time step all dynamical equations (including the nonspherical part) are solved. The value of the vapor pressure is that corresponding to the bubble wall temperature computed at the preceding time step using equation (32). Figure 3 shows an example of the results obtained with the developed numerical code for the case of \( N \) symmetrically located bubbles on a spherical shell. Both, the motion of a characteristic point of the bubble wall and the bubble wall temperature are plotted versus time. The motion of different points on the bubble wall is not always the same, but space limitations do not allow us to elaborate here (see [5] and [9] for details). Basically one observes a flattening of the bubble and its elongation in the direction tangential to the sphere. The qualitative relative comparison of the cases of various numbers of bubbles, \( N \), is however always the same for any point on the bubble surface. Figure 3 reconfirms the results of Fig. 1 on the inhibiting effect of the presence of interacting bubbles on the bubble growth rate. This result is important for any heat exchange prediction and is reflected by the bubble wall temperatures which are shown to decrease slower in the presence of the other bubbles. The comparison of the growth of a bubble in a superheated liquid to that in a cold liquid is satisfactory, both showing the known asymptotic behavior. The results concerning the last phase of the growth of multiple bubbles, namely collapse instead of growth and consequently temperature surge, are, however, doubtful and instead reflect the expected failure of the asymptotic method in describing correctly the bubble behavior when the two-phase medium becomes dense. As we have concluded in the case of inertia-controlled bubble clouds [7], this failure is probably due to the compressibility of the bubbly medium which was not taken into account. Indeed, the propagation of the information from one bubble to another takes place, in fact, in a finite time due to the finite wave speed in the medium. As a result, the effects of the various bubbles add up with retarded times. Once this is accounted for, the cumulative modification of the pressure field surrounding a particular bubble will be significantly reduced. Another source of method failure is inherent in the growth phenomenon since the ratio of bubble radius to inter-bubble distance constantly increases, necessarily inducing nonvalidity of an asymptotic approach beyond a time limit.

Acknowledgment

This work was supported by a National Science Foundation Grant No. MEA-8260689.

References