Optimum Configuration of an Expanding-contracting-nozzle for Thrust Enhancement by Bubble Injection

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ABSTRACT

This paper addresses the concept of thrust augmentation through bubble injection into an expanding-contracting nozzle. Two-phase models for bubbly flow in an expanding-contracting nozzle are developed, in parallel with laboratory experiments and used to ascertain the geometry configuration for the nozzle that would lead to maximum thrust enhancement upon bubble injection.

For preliminary optimization of experimental setup's design, a quasi 1-D approach is used. Averaged flow quantities (such as velocities, pressures, and void fractions) in a cross-section are used for the analysis. The mixture continuity and momentum equations are numerically solved simultaneously, along with equations for bubble dynamics, bubble motion, and an equation for conservation of the total bubble number. Various geometric parameters such as the exit and inlet areas, the area of the bubble injection section, the presence of a throat and its location, the length of the diffuser section and the length of the contraction section are varied, and their effects on thrust enhancement are studied. Investigation on the effect of the injected void fraction is also carried out. The key objective function of the optimization is the normalized thrust parameter, which is the thrust with bubble injection minus the thrust with liquid only divided by the inlet liquid momentum.

An approximate analytical expression for the normalized thrust parameter was also derived starting from the mixture continuity and momentum equations. This analytical expression involved flow variables only at three locations; inlet section, injection section, and outlet section, and the expression is simple enough to produce a quick concept design of the diffuser-nozzle thruster. The numerical and analytical approaches are verified against each other and the limitations of the analytical approach are discussed.
NOMENCLATURE

\( A \) : Cross-sectional area

\( C \) : Ratio of the exit area to the inlet area (dimensionless)

\( c_b \) : Sound speed in the liquid at bubble surface

\( C_d \) : Drag coefficient (dimensionless)

\( C_f \) : Friction coefficient (dimensionless)

\( C_{inj} \) : Ratio of the exit area to area of the injection section (dimensionless)

\( k \) : Polytropic gas constant (dimensionless)

\( N \) : Number of bubbles per unit volume

\( p \) : Pressure

\( p_l \) : Pressure in the liquid at the bubble wall

\( p_g \) : Bubble gas pressure

\( p_v \) : Vapor pressure

\( R \) : Bubble radius

\( R_{exit} \) : Exit radius of the expanding-contracting nozzle

\( R_{inj} \) : Radius of injection section of the nozzle

\( R_{inlet} \) : Inlet radius of the expanding-contracting nozzle

\( u_b \) : Local bubble velocity

\( u_m \) : Mixture velocity

\( \alpha \) : Void fraction (dimensionless)

\( \alpha_0 \) : Injected void fraction (dimensionless)

\( \beta \) : Volume fraction of liquid \((1 - \alpha)\) (dimensionless)

\( \beta_0 \) : Volume fraction of liquid at injection \((1 - \alpha_0)\) (dimensionless)
\[
\gamma : \text{Surface tension parameter}
\]
\[
\mu : \text{Liquid viscosity}
\]
\[
\rho_l : \text{Liquid density}
\]
\[
\rho_m : \text{Mixture density}
\]
\[
\xi_m : \text{Normalized thrust parameter (dimensionless)}
\]

INTRODUCTION

Recent studies [1, 2] have indicated that bubble injection in a waterjet can significantly improve the net thrust and overall propulsion efficiency. The importance of this concept was attributed to the fact that bubble augmented thrust can be achieved even at very high vehicle speeds unlike traditional propulsion devices.

Various analytical, numerical and experimental efforts have been carried out since to formalize the concept of bubble augmented thrust [3, 4], and a number of prototypes have been built for experimental verification of this concept [5, 6]. In most of these experimental set-ups, the liquid enters a ramjet where it is compressed first by passing through a diffuser (ram effect), then pressurized gas is injected into the liquid via mixing ports, which acts like the energy source of the ramjet. The multiphase mixture is then accelerated through a converging nozzle.

To obtain a first approximation of the mixture flow field inside the air augmented nozzle numerically, one can use classical quasi-one dimensional analyses [7-11]. In such approaches, flow quantities are assumed to be uniform in a direction perpendicular to the axis of the nozzle (i.e. only quantities averaged over a cross-section are used). Early models [12] often neglected the relative motion between the bubbles and the liquid. These effects, however, were later included to account for the inertial effects on bubble dynamics [13]. Furthermore, a more advanced model has been proposed to characterize the bubble dynamics using the Rayleigh-Plesset equation [8-10]. By capturing inertial effects, this model is able to represent transient shock waves in a bubbly mixture. A few recent works [7, 14] have used similar models for analyses of bubbly flows in converging diverging nozzles.

Thrust enhancement demonstrations require further modeling and controlled experimentation as we are presently undertaking at DYNAFLOW. One of the primary objectives of our study is to optimize the nozzle geometry
for an expanding-contracting nozzle, BAP\textsuperscript{1} (Figure 1), in order to obtain good thrust enhancement. In the present paper, two modeling approaches are described. The results from these models serve as good starting guidelines for experimentation.

![Figure 1. Expanding-contracting Nozzle.](image)

The first approach (1-D BAP) is a one-dimensional numerical technique for solving the mixture flow in the nozzle. The second approach (0-D BAP) is purely analytical – derived starting from the mixture continuity and momentum equations in the expanding-contacting nozzle. Even though the second approach has certain limitations, it is very useful in verifying the results from the first approach for simple cases and also for generating quick concept designs.

**NUMERICAL PROCEDURE (1-D BAP MIXTURE MODEL)**

The unsteady 1-D flow of a bubbly mixture through a nozzle of varying cross-section area, \( A(x) \), can be described by:

\[
\begin{align*}
\frac{\partial \rho_m}{\partial t} + \frac{1}{A} \frac{\partial (\rho_m u_m A)}{\partial x} &= 0, \\
\frac{\partial \rho_m u_m}{\partial t} + \frac{1}{A} \frac{\partial (\rho_m u_m A u_m)}{\partial x} + \frac{\partial p}{\partial x} &= 0,
\end{align*}
\]

\(1\)

\textsuperscript{1} BAP: Bubble Augmented Propulsor
where \( \rho_m \), \( u_m \), and \( p \) are the mixture density, velocity and pressure. If we consider the case where the bubbles are injected as a continuous stream, and look for the steady state solution of the cross-section averaged flow quantities, (1) reduces to the following form:

\[
\frac{\partial (\rho_m u_m A)}{\partial x} = 0,
\]

\[
\frac{1}{A} \frac{\partial (\rho_m u_m A u_m)}{\partial x} + \frac{\partial p}{\partial x} = 0,
\]

(2)

with \( \rho_m \), \( u_m \), \( A \), and \( p \) all being functions only of \( x \).

In this study, the liquid is assumed to be incompressible, i.e. all compressibility effects of the mixture arise from the disperse gas phase only. The mixture density in terms of the liquid density, \( \rho_l \), the gas density, \( \rho_g \), and the void fraction, \( \alpha \) is:

\[
\rho_m = \rho_l (1 - \alpha) + \alpha \rho_g.
\]

(3)

By noting that the density of the injected air/gas (\( \rho_g \sim 10^{-3} \text{ Kg/l} \)) is very small compared to the density of the surrounding water (\( \rho_l \sim 1 \text{ Kg/l} \)), we can express the mixture density in terms of the liquid density, \( \rho_l \) and the void fraction, \( \alpha \), only and numerically ignore the gas density term contribution:

\[
\rho_m = \rho_l (1 - \alpha).
\]

(4)

Note that the void fraction, \( \alpha \), is the volume occupied by the bubbles per unit mixture volume, and can be written using the number of bubbles of radius \( R_i \) per unit volume, \( N_i(R_i) \):

\[
\alpha = \frac{4}{3} \pi \sum_i R_i^3 N_i(R_i).
\]

(5)

In the present analysis we consider input of mono-disperse bubbles such that:

\[
\alpha(x) = \frac{4}{3} \pi R_i^3(x) N(x).
\]

(6)

Furthermore, in this paper, we assume that other than at the injection location, no bubbles are created or destroyed, that is the total number of bubbles in the domain in the steady state solution is invariant (the total number
of bubbles that enters and exits the computational domain is the same). The corresponding equation of conservation of the number of bubbles can be obtained by writing conservation of the bubble number in an elementary domain length, \( dx \), and has already been derived in textbooks [15] as follows:

\[
\frac{\partial N}{\partial t} + \frac{1}{A} \frac{\partial (u_b AN)}{\partial x} = 0, \tag{7}
\]

where, \( u_b \) is the local bubble velocity. This reduces (under the quasi steady assumption) to the following form:

\[
\frac{\partial}{\partial x}(u_b AN) = 0. \tag{8}
\]

To describe the dynamics of a local bubble, we assume that the bubbles are spherical and that behavior is governed by the modified Keller-Herring equation [16, 18]:

\[
(1 - \frac{\dot{R}}{c_b})R\ddot{R} + \frac{3}{2} (1 - \frac{\dot{R}}{3c_b})\dot{R} = \left(\frac{u_m - u_b}{4}\right)^2 + \frac{1}{\rho_m} (1 + \frac{\dot{R}}{c_b} + \frac{R}{c_b} \frac{d}{dt}) \left[ p_v + p_g - p - \frac{2\gamma}{R} - 4\mu \frac{\dot{R}}{R} \right]. \tag{9}
\]

In the above equation, \( R \) is the spherical bubble radius, dots represent time derivatives, and \( c_b \) is the sound speed in the liquid at the bubble surface. Equation (9) is commonly used and describes nonlinear bubble oscillations in a slightly compressible liquid valid when the bubble wall speed is small compared to the sound speed in the liquid. It is a first order improvement over the classical Rayleigh Plesset [17, 26] equation, which ignores the liquid compressibility. In this paper, for all cases considered the bubble growth rate (radial bubble wall speed) is very much smaller than the water sound speed, which justifies the use of (9).

Equation (9) also assumes a balance of normal stresses at the bubble wall:

\[
p_l = p_v + p_g - \frac{2\gamma}{R} - 4\mu \frac{\dot{R}}{R}, \tag{10}
\]

\[
\]
where \( p_l \) is the pressure in the liquid at the bubble wall, \( p_g \) is the bubble gas pressure, \( p_v \) is the liquid vapor pressure, \( \gamma \) is the surface tension parameter and \( \mu \) is the dynamic viscosity of the liquid. Since gas diffusion is very slow, we assume that the mass of gas inside each bubble does not change and that the bubble follows a polytropic compression law:

\[
p_g = p_{g0} \left( \frac{R_0}{R} \right)^{3k},
\]

where \( k \) is the polytropic gas constant, \( p_{g0} \) is the initial bubble gas pressure, and \( R_0 \) is the corresponding initial bubble radius.

In the numerical simulations in this paper, we assumed adiabatic compression of the air bubbles and used \( k = 1.4 \) and a liquid speed of sound of 1,500 m/s.

The local bubble translation velocity is obtained by solving the following simple equation for bubble motion [19], even though more complex equations exist [20, 21, 22]:

\[
\frac{du_b}{dt} = -\frac{3}{\rho_l} \frac{dp}{dx} + \frac{3}{4} (u_m - u_b) \frac{|u_m - u_b|}{R} \frac{Cd}{R} + 3 \frac{\dot{R}}{R} (u_m - u_b).
\]

Equation (12) considers the forces due to the liquid added mass, the pressure gradient, a drag force (the second term on RHS), and a term proportional to the bubble wall velocity (the last term on RHS), which couples the bubble breathing mode with its translation velocity. The added mass term is combined with the pressure term (the first term on RHS) where the factor 3 appears. No lift forces are included due to of the fact that the study is limited to a 1D motion approximation and the history term (Basset term) [20, 21, 22] is neglected since we are considering a steady time independent solution and previous studies have concluded in that this term is of a secondary order when compared to the other forces [22]. Gravity forces are not included because we consider here a horizontal problem 1D problem.
The drag coefficient, \( C_d \), in equation (12) is that of a spherical contaminated bubble and used the following model for the drag coefficient [23]:

\[
C_d = \frac{24}{R_{eb}} \left( 1 + 0.197 R_{eb}^{0.63} + 2.6 \times 10^{-4} R_{eb}^{1.38} \right);
\]

\[
R_{eb} = \frac{2 \rho_m R |u_m - u_b|}{\mu}.
\] (13)

Since our approach is quasi-steady, the time as we follow the bubble dynamics is transformed into the bubble dynamics along the axial coordinate. The following space-time transformation is applied for each bubble in order to convert the time derivatives into derivatives along the axial direction:

\[
\frac{\partial}{\partial t} = u_b \frac{\partial}{\partial x}.
\] (14)

Therefore, the time derivatives in Equation (9) and Equation (12) are transformed to derivatives with respect to \( x \) using (14), to give:

\[
u_b \frac{du_b}{dx} + \frac{3}{\rho_1} \frac{dp}{dx} = \frac{3}{4} (u_m - u_b) \frac{|u_m - u_b|}{R} + 3 \frac{\dot{R}}{R} (u_m - u_b),
\]

\[
\dot{R} = u_b \frac{dR}{dx}.
\] (15)

The set of equations, (2), (6), (8), (9), (12) and (14) is then integrated in space using an embedded adaptive stepping explicit Runge-Kutta integration. The difference between the solutions of the 4th and the 5th order accurate schemes is taken as a measure of the integration error and is used to control the forward step size. In addition to the above explicit scheme, an implicit scheme was used for the numerical integration, which made the convergence more robust. The inlet velocity and the exit pressure serve as two convenient boundary conditions.
ANALYTICAL APPROACH (0-D BAP MIXTURE MODEL)

The axisymmetric expanding-contracting nozzle geometry can be defined by the following three cross sectional dimensions (Figure 1):

- Inlet Radius: $R_{inlet}$
- Injection Radius: $R_{inj}$
- Exit Radius: $R_{exit}$

Based on these dimensions, the following contraction area ratios between these sections can be defined:

$$C_{inj} = \left( \frac{R_{exit}}{R_{inj}} \right)^2, \quad C = \left( \frac{R_{exit}}{R_{inlet}} \right)^2,$$

(16)

The velocity and pressure are defined at the three locations as well:

- Velocity and pressure at the inlet: $V_{inlet}$ and $p_{inlet}$,
- Velocity and pressure at the injection section: $V_{inj}$ and $p_{inj}$,
- Velocity and pressure at the exit: $V_{exit}$ and $p_{exit}$.

Continuity

When there is no bubble injection, the mass conservation requires:

$$\rho_l R_{inlet}^2 V_{inlet} = \rho_l R_{inj}^2 V_{inj,0} = \rho_l R_{exit}^2 V_{exit,0},$$

(17)

where the index 0 implies pure liquid. This index is not used at the inlet because we assume that there we always have a pure liquid.

After injection, if we think of the mass flow rate in terms of mixture density and mixture velocity, conservation of mass gives us the following relation:
where the index $\alpha$ indicates that the fluid is a mixture. Expressing the mixture density at the injection location and at the exit ($\rho_{m,\text{inj}}$, $\rho_{m,\text{exit}}$) in terms of the liquid density and void fraction (4), we have, after simplification:

$$R_{\text{inlet}}^2 V_{\text{inlet}} = R_{\text{inj}}^2 \rho_{m,\text{inj}} V_{\text{inj},\alpha} = R_{\text{exit}}^2 \rho_{m,\text{exit}} V_{\text{exit},\alpha},$$  \hspace{1cm} (18)

where $\alpha_0$ is the injected void fraction and $\alpha$ is the void fraction at the exit. For convenience, we define:

$$\beta_0 = 1 - \alpha_0, \quad \beta = 1 - \alpha.$$  \hspace{1cm} (20)

In our 0-D analysis, we assume no slip between phases i.e. the mixture velocity is same as the liquid velocity at these locations.

### Bernoulli Equation

When there is no injection, using energy conservation (Bernoulli’s equation), we have in the pure liquid:

$$p_{\text{inlet},0} + \frac{\rho_l}{2} V_{\text{inlet}}^2 = p_{\text{inj},0} + \frac{\rho_l}{2} V_{\text{inj},0}^2 = p_{\text{exit}} + \frac{\rho_l}{2} V_{\text{exit},0}^2.$$  \hspace{1cm} (21)

If there is injection, we write the Bernoulli equation separately upstream and downstream of the injection.

Applying Bernoulli’s equation between the inlet and a point immediately before the injection location (subscript $-$), and ignoring the gas input energy ($\rho_g \cdot \rho_l$), we have:

$$p_{\text{inlet},\alpha} + \frac{\rho_l}{2} V_{\text{inlet}}^2 = p_{\text{inj},\alpha} + \frac{\rho_l}{2} V_{\text{inj},\alpha}^2.$$  \hspace{1cm} (22)
Applying Bernoulli equation between a point immediately after the injection location (subscript \(+\)) and the exit, and ignoring again the gas energy, we have:

\[
p_{\text{inj}^+,\alpha} + \frac{\rho_l}{2} (1 - \alpha_o) V_{\text{inj}^+,\alpha} = p_{\text{exit}} + \frac{\rho_l}{2} (1 - \alpha) V_{\text{exit},\alpha}^2.
\]  

(23)

where the \((1-\alpha_o)\) and \((1-\alpha)\) terms multiplied by \(\rho_l\) account for the mixture density.

Applying mass conservation (ignoring air mass) between a point just before the injection (subscript \(\sim\)) and a point just after the injection (subscript \(+\)), we have:

\[
\rho_l V_{\text{inj}^-} = \rho_l (1 - \alpha_o) V_{\text{inj}^+}.
\]

(24)

or

\[
V_{\text{inj}^+} = V_{\text{inj}^-} / (1 - \alpha_o).
\]

(25)

Applying momentum balance (ignoring air mass) between a point just before the injection location (subscript \(\sim\)) and a point just after the injection location (subscript \(+\)), we have:

\[
(p_{\text{inj}^-} + \rho_l V_{\text{inj}^-}^2) \pi R_{\text{inj}}^2 = (p_{\text{inj}^+} + \rho_l (1 - \alpha_o) V_{\text{inj}^+}^2) \pi R_{\text{inj}}^2.
\]

(26)

Thus, the pressure jump across the injection location is given as follows:

\[
\Delta p = p_{\text{inj}^-} - p_{\text{inj}^+} = \rho_l V_{\text{inj}^-}^2 \frac{\alpha_o}{1 - \alpha_o} = \rho_l V_{\text{inj}^+}^2 \alpha_o (1 - \alpha_o).
\]

(27)
**Relationship between Inlet Pressure and Exit Pressure**

We now construct the overall solution of the problem using the above equations. This will lead us to a relation connecting the inlet pressure and the exit pressure.

Using (17) and (19), we have:

\[ V_{exit,0} = C^{-1}V_{inlet} \]  \hspace{1cm} (28)

\[ V_{exit,\alpha} = \frac{C^{-1}}{1-\alpha}V_{inlet} \]  \hspace{1cm} (29)

From continuity between a point after the injection location and the exit (using Eqn.(20)):

\[ V_{inj^+,\alpha} = C_{inj}V_{exit,\alpha}(\beta / \beta_0) \]  \hspace{1cm} (30)

From continuity between a point before the injection location and the inlet:

\[ V_{inj^-,\alpha} = C_{inj}C^{-1}V_{inlet} \]  \hspace{1cm} (31)

Using (22) to (27) in order to relate the inlet pressure and the exit pressure, we have:

\[ p_{inlet,\alpha} = p_{inj^-,\alpha} + \frac{\rho_l}{2} \left( V_{inj^-,\alpha}^2 - V_{inlet}^2 \right) \]

\[ = p_{inj^+,\alpha} + \Delta p + \frac{\rho_l}{2} \left( V_{inj^+,\alpha}^2 - V_{inlet}^2 \right) \]

\[ = p_{exit} + \frac{\rho_l}{2} \left( \beta V_{exit,\alpha} - \beta_0 V_{inj^+,\alpha} \right) \]

\[ + \Delta p + \frac{\rho_l}{2} \left( V_{inj^-,\alpha}^2 - V_{inlet}^2 \right) \]  \hspace{1cm} (32)

Using (30) and (31), (32) becomes:
Replacing $\Delta p$ from (27), we have:

$$P_{inlet,\alpha} = P_{exit} + \frac{\rho_l}{2} \left[ \beta_0 \left( \frac{\beta_0}{\beta} C_{inj}^{-2} - 1 \right) V_{inj}^{2,\alpha} \right] + \frac{\rho_l}{2} \left[ \frac{2\alpha_o}{\beta_0} V_{inj}^{2,\alpha} + (C_{inj}^{2} C^{-2} - 1) V_{inlet}^{2} \right].$$

Expressing $V_{inj}^{2,\alpha}$ in terms of $V_{inj}^{-\alpha}$, by using (25), we get:

$$P_{inlet,\alpha} = P_{exit} + \frac{\rho_l}{2} \left[ \frac{\beta_0}{\beta} C_{inj}^{-2} - 1 \right] V_{inj}^{2,\alpha} + \frac{2\alpha_o}{\beta_0} V_{inj}^{2,\alpha} + \left( C_{inj}^{2} C^{-2} - 1 \right) V_{inlet}^{2}.$$  \hspace{1cm} (35)

Expressing $V_{inj}$ in terms of $V_{inlet}$, by using (31), and simplifying, we get:

$$P_{inlet,\alpha} = P_{exit} + \frac{\rho_l}{2} \left[ \frac{\beta_0}{\beta} C_{inj}^{-2} - 1 \right] V_{inlet}^{2} - \frac{2\alpha_o}{\beta_0} V_{inlet}^{2} + \left( C_{inj}^{2} C^{-2} - 1 \right) V_{inlet}^{2}.$$  \hspace{1cm} (36)

The above equation is an important approximate relation between the inlet pressure and the exit pressure and is intended to be used as a first order estimation for design study purposes. The limit of this simplified approach will be discussed further later through comparison with more precise calculations.

**Normalized Thrust Increase**
The thrust (without bubble injection) of the expanding-contracting nozzle with circular cross sections is given by:

\[
T_R = \pi \left[ p_{exit}R_{exit}^2 - p_{inlet,0}R_{inlet}^2 + \rho_l R_{exit}^2 V_{exit,0}^2 - \rho_l R_{inlet}^2 V_{inlet}^2 \right].
\]  (37)

When there is air injection, the thrust of the expanding-contracting nozzle can be given as follows:

\[
T_{R,\alpha} = \pi \left[ p_{exit}R_{exit}^2 - p_{inlet,\alpha}R_{inlet}^2 + \rho_l R_{exit}^2 (1-\alpha)V_{exit,\alpha}^2 - \rho_l R_{inlet}^2 V_{inlet}^2 \right].
\]  (38)

Let us define the normalized thrust gain parameter as follows:

\[
\xi_m = \frac{T_{R,\alpha} - T_R}{T_{m-inlet}} = \frac{T_{R,\alpha} - T_R}{\pi \rho_l R_{inlet}^2 V_{inlet}^2},
\]  (39)

where \(T_{m-inlet}\) is the inlet momentum rate.

Using expressions (37) and (38), we have:

\[
\xi_m = \frac{-p_{inlet,\alpha} + \rho_l C(1-\alpha)V_{exit,\alpha}^2 + p_{inlet,0} - \rho_l CV_{exit,0}^2}{\rho_l V_{inlet}^2}.
\]  (40)

Using (20), and expressing all velocities in terms of the inlet velocity [(28) and (29)], we have:

\[
\xi_m = \frac{(p_{inlet,0} - p_{inlet,\alpha}) + \rho_l CV_{inlet}^2 \left[ C^{-2} \beta^{-1} - C^{-2} \right]}{\rho_l V_{inlet}^2}.
\]  (41)

Using equations (21) and(36), we have:
\[ P_{\text{inlet,0}} - P_{\text{inlet,} \alpha} = \frac{p_1}{2} (C^{-2} - 1) V_{\text{inlet}}^2 - \frac{p_1}{2} \left[ \frac{(\beta_0 / \beta + \alpha_o C_{\text{inj}}^2)}{(1 - \alpha_o) C^2} - 1 \right] V_{\text{inlet}}^2. \] 

Therefore, equation (41) becomes:

\[ \xi_m = \frac{1}{2} C^{-2} \left[ 1 - \left( \frac{\beta_0}{\beta} + \alpha_o C_{\text{inj}}^2 \right) \beta_o^{-1} \right] + C^{-1} \left( \beta^{-1} - 1 \right) \] 

### Relationship between Exit Void Fraction and Injected Void Fraction

An equation relating the injected void fraction, \( \alpha_o \), to the exit void fraction, \( \alpha \), can be obtained by using the equation of state for a two-component bubbly medium [26]. If we assume a polytropic compression law for the gas and incompressible liquid phase, the pressure can be expressed as:

\[ \frac{p}{p_o} = \left[ \frac{\alpha_0}{1 - \alpha_0} \right]^k, \] 

using the variables at the reference condition, \( p_o \) and \( \alpha_o \). A similar relation for a compressible liquid assuming \( p_o = \text{constant} \) can be found in [26]:

\[ \left( \frac{p}{p_o} \right)^{k + 1} \rho \rho_c^2 \left( \frac{p}{p_o} \right)^k \left[ \frac{\alpha_0}{1 - \alpha_0} + \frac{k}{k + 1} \frac{p_o}{\rho \rho_c^2} \right] = -\alpha. \] 

This equation describes the void fraction changes due both the compression of the liquid and of the gas in the bubbles. In deriving the above relation, it was assumed that:

- the density of the gas, \( \rho_g \), is negligible relative to the density of the liquid, \( \rho_l \),
- the disperse gas phase obeys a polytropic compression law,
- surface tension is ignored at the interface of the two components (\( p \) varies smoothly in the medium) and
- the void fraction, \( \alpha \), is small.

We take the reference pressure as \( p_o = p_{\text{inj} + \alpha} \) since our \( \alpha_o \) is defined at the injection location. Using (45), the exit pressure can be related to the reference pressure at the injection by:
\[ \alpha = \left[ \frac{\alpha_0}{1-\alpha_0} + \frac{k}{k+1} \frac{p_{\text{inj},\alpha}}{p_{\text{exi}}} \right] \left( \frac{p_{\text{inj},\alpha}}{p_{\text{exi}}} \right)^{\frac{1}{k}} - \frac{k}{k+1} \frac{p_{\text{exi}}}{\rho_l c_l^2}. \]  

(46)

\[ p_{\text{inj},\alpha} = p_{\text{exi}} + \frac{p_l}{2} \left[ \beta V_{\text{exi},\alpha} - \beta_0 V_{\text{inj},\alpha}^2 \right]. \]  

(47)

Using (30) in the above equation to replace \( V_{\text{inj},\alpha} \), we have:

\[ p_{\text{inj},\alpha} = p_{\text{exi}} + \frac{p_l}{2} V_{\text{exi},\alpha} \beta \left[ 1 - \beta_0^{-1} \beta C_{\text{inj}}^2 \right] \]  

(48)

Using (29) in the above equation, we have:

\[ p_{\text{inj},\alpha} = p_{\text{exi}} + \frac{p_l}{2} V_{\text{inlet}}^2 \beta^{-1} \left[ 1 - \frac{\beta}{\beta_0} C_{\text{inj}}^2 \right] \]  

(49)

Finally, Equation (46) and Equation (49) can be solved iteratively to obtain \( \alpha \) (or \( \beta \)).

**RESULTS AND DISCUSSION**

**Validation of 1-D BAP**

Independent studies [3] were conducted at DYNAFLOW using 1-D BAP for a particular expanding-contracting nozzle configuration and the results were verified against results from our discrete bubble model (DBM) 3DYNAFS_DSM© coupled with the 3D Navier-Stokes viscous solver, 3DYNAFS_Vis© and with FLUENT [24, 25]. For illustration, Figure 2 compares the total thrust predicted by the 1-D BAP with the results from 3DYNAFS_DSM© coupled with Fluent. The figure also shows the results of 1-D BAP when a static balance of the gas pressures in the bubbles is considered instead of fully solving the bubble dynamics equation, (9). In this case, all three agree well up to a void fraction of 0.4, above which both 1-D models start to deviate from the full 3D predictions. We believe the difference came from the bubble slip velocity which is modeled more directly in the 3D model, 3DYNAFS_DSM©. The 1D model also agreed well with our recent experimental measurements described in [4].
Dependence of the Normalized Thrust Parameter on the Contraction Area Ratio \( C \)

The analytical expression for \( \xi_m \) (obtained using 0-D BAP) was compared against results from the 1-D BAP code. Figure 3 shows the length dimensions of the expanding-contracting nozzle under study. The 1-D BAP runs were carried out for different values of the ratio of the exit area to the inlet area, \( C \), defined in (16). The runs were carried out using two inlet velocities of 2.4 m/s and 3.57 m/s, an exit pressure of 101,325 Pa and a \( R_{\text{ej}} \) of 9.35 cm.

Figure 4 shows the variations of the normalized thrust parameter (\( \xi_m \)) with variation of \( C \) (varying \( R_{\text{ej}} \)) for an inlet velocity of 2.4 m/s. The curves have an optimum close to \( C=1 \). Also, as expected, the normalized thrust parameter is higher for higher injected void fraction. It must be noted that the analytical expression compares better with the numerical results when the injected void fraction is small. Figure 5 shows a similar set of results for an inlet velocity of 3.57 m/s. The 1-D BAP curves for these two inlet velocities are almost identical. The analytical results, on the other hand, are slightly different for the two cases. For a higher inlet velocity, the analytical approach predicts much higher thrust gain. This is because the computed exit void fraction [from Equation (46)] is significantly higher than the injected void fraction when the inlet velocity is large. Even though this effect is expected in reality, 1-D BAP results suggest that the 0-D model over-predicts the thrust gain for larger inlet velocities. As mentioned earlier 1-D BAP shows similar trends, i.e. the thrust gain increases when the inlet velocity increases (Figure 6). However,
the inlet velocity must be large (10 m/s for the case shown in Figure 6) to see significant improvement in thrust gain.

![Diagram of expanding-contracting nozzle dimensions.](image)

*Figure 3. Dimensions of the expanding-contracting nozzle.*

![Graph showing normalized thrust parameter versus C.](image)

*Figure 4. Normalized thrust parameter versus C (V_{inlet} = 2.4 m/s). Analytical expression for \( \xi_{in} \) is compared against results from the 1-D BAP code.*
Figure 5. Normalized thrust parameter versus $C$ ($V_{\text{inlet}}=3.57$ m/s).
Analytical expression for $\tilde{\xi}_m$ is compared against results from the 1-D BAP code.

Figure 6. Effect of increasing the inlet velocity on the normalized thrust parameter: 1-D BAP.

In addition, one can observe from equation (43) that for $C=1$, $\tilde{\xi}_m \approx \alpha_0 / (1 - \alpha_0) \times \left[ (1 - C_{\text{inj}}^2) / 2 \right]$ (assuming $\alpha = \alpha_0$). Figure 7 shows the variation of $\tilde{\xi}_m$ with variation of the injected void fraction $\alpha_0$ as obtained from the two approaches (for $C=1$). The results reaffirm the fact that increasing the injection void fraction increases thrust gain.
Dependence of the Normalized Thrust Parameter on the Contraction Area Ratio $C_{inj}$

Physical intuition suggests that increasing the ratio of the area at the injection location to the exit area would increase the normalized thrust gain. This is because, increasing the propulsor area where the air is injected would imply that the bubbles are injected at a higher flow pressure and their expansion at the exit (lower pressure) would prominently cause the liquid at the exit to flow out with higher velocity, contributing to higher thrust.

Figure 8 and Figure 9 capture the variation of $\xi_m$ with $1/C_{inj}$ for two different values of $\alpha_0$. In these runs, $C$ is fixed at 1.1 (close to optimum). It appears that increasing the area of the injection region does not have significant effect on the normalized thrust parameter beyond a certain limit. For instance, when $\alpha_0=0.05$, $\xi_m$ for $1/C_{inj}=6$ is approximately 95% of the value of $\xi_m$ for $1/C_{inj}=12$.

Again, the 0-D BAP captures the trend better when the injected void fraction is small.
Figure 8. Normalized thrust parameter versus $1/C_{\text{inj}} - \alpha_0 = 0.05$, $C=1.1$.

Figure 9. Normalized thrust parameter versus $1/C_{\text{inj}} - \alpha_0 = 0.10$, $C=1.1$.

Presence of a Throat

After establishing that the normalized thrust parameter is optimum for $C \leq 1$, we studied the effect of the presence of a throat just before the exit of the expanding-contracting nozzle. Starting with a base configuration and
$C=1$, Figure 10a, two modifications are applied. The first modification involves reducing the length of the contraction section, Figure 10b, and the second modification involves extending the length of the contraction section, Figure 10c, in order to incorporate a throat. It must be noted that both these modifications ensure that $C=1$.

Within the second modification, different configurations were tried. These configurations, essentially, determine the size of the throat (Figure 11). Configuration 1 has no throat [dashed line in Figure 10 (c)] while Configuration 10 has a very narrow throat.

![Geometry of the expanding-contracting nozzle.](image-url)
Figure 11. Different configurations (expanding-contracting nozzle) studied.

Figure 12 shows the variation of the normalized thrust parameter, $\xi_m$, with variation of the injected void fraction, and Figure 13 shows the variation of the total thrust versus variation of $\alpha_0$ for an inlet velocity of 2.4 m/s.

Since the 0-D BAP analytical approach only involves the area ratios $C$ and $C_{inj}$ and does not include the geometry variation occurring in-between, it cannot capture the effect of the presence of a throat before the exit. The results from 1-D BAP with a cut-exit match the 0-D BAP results quite well (Figure 12).

From the 1-D BAP results, it appears that introducing a throat is not beneficial in increasing thrust gain. This is somewhat contrary to physical intuition. One would expect the presence of a throat to accelerate the liquid (due to bubble expansion) and consequently improve thrust gain. In order to confirm the trend for other flow conditions, a simulation with a higher inlet velocity was performed. It was thought that having higher flow velocity would lead to a higher void fraction in the throat region and would consequently increase thrust gain. Figure 14 shows the variation of the normalized thrust parameter, $\xi_m$, with the variation of the injected void fraction for
selected configurations at the inlet velocity of 10m/s. Even in this case, presence of a throat did not help in achieving higher thrust gain.

**Figure 12.** Normalized thrust parameter versus injected void fraction (different configurations, C=1, $V_{inlet}=2.4$ m/s).

**Figure 13.** Total thrust versus injected void fraction (different configurations, C=1, $V_{inlet}=2.4$ m/s).
CONCLUSION

One of the primary objectives of this study was to optimize the geometry of an expanding-contracting nozzle in order to obtain high thrust enhancement. To achieve this, two modeling approaches were followed. The first approach (1-D BAP) used a quasi-one-dimensional numerical technique for solving the mixture flow in the nozzle. The second approach (0-D BAP) involved analytical derivation of the normalized thrust parameter.

![Figure 14. Normalized thrust parameter versus injected void fraction (different configurations, C=1, V_{inlet}=10 m/s).](image)

The analytical approach is very useful in providing physical understanding of the bubbly flow phenomenon in the nozzle, verifying the results of the first approach, and generating quick concept designs. On the other hand, the first approach, with the use of numerical solutions, can predict the nozzle performance more accurately with more detailed considerations including the followings:

1) The approach accounts for the dynamics of individual bubbles in the mixture.
2) The approach allows slip between the bubble and the mixture flow around.
3) The approach includes the geometric variation along the nozzle including multiple contractions. Hence, it can capture the effect of a throat if exists.
An even further complete approach involving 3D computation coupling a viscous Reynolds-Averaged Navier-Stokes solver on Eulerian grids and a Lagrangian tracking of discrete bubbles is described elsewhere [3, 4, 24, 25]. For simple configurations, it was found that the maximum normalized thrust gain occurs when the exit area is equal to the inlet area. Additionally, the thrust gain is higher for higher injected void fraction. It was also found that increasing the area of the injection region (relative to the exit) increases the normalized thrust gain to a certain extent.

Moreover, the 1-D BAP simulations indicated that the presence of a throat under the conditions we have studied reduces thrust gain. This conclusion should be reexamined with further analysis and experiments since the 1-D BAP code does not capture the complex flow physics (e.g. separation, compressibility of the mixture, non-uniformity in radial direction, etc.) that would prevail in the presence of a throat.

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REFERENCES


