ABSTRACT

Background: Bubbly flows are used in a wide variety of applications and require accurate modeling. In this paper three modeling approaches are investigated using the geometrically simple configuration of a gas bubble strongly oscillating in a bubbly medium.

Method of Approach: A coupled Eulerian-Lagrangian, a multi-component compressible, and an analytical approach are compared for different void fractions.

Results: While the homogeneous mixture models (analytical and multi-component) compare well with each other, the Eulerian-Lagrangian model captures additional features and inhomogeneities. The discrete bubbles appear to introduce localized perturbations in the void fraction and the pressure distributions not captured by homogeneous mixture models.

Conclusion: The bubbly mixture impedes the growth and collapse of the primary bubble while wavy patterns in the velocity, pressure, and void fraction fields propagate in space and time.

KEYWORDS: Multiphase Flow, Bubble dynamics, Bubbly Flow

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1. INTRODUCTION

Two-phase bubbly flows are usually modeled using an equivalent homogeneous continuum, Eulerian two fluid approaches, or Eulerian-Lagrangian approaches wherein the bubbles are treated as discrete particles. Homogeneous models are useful for low void fraction flows, whereas Eulerian-Lagrangian approaches are more appropriate for higher void fractions [1-11]. For applications involving large bubble volume changes such as cavitation or shock and pressure wave propagation, the bubbles’ dynamics, relative motions, and deformations, and their effects on the bubbly flow are essential and require detailed multi-scale descriptions. In this paper we consider a geometrically simple but challenging numerical configuration for comparing three methods: a continuum homogeneous model, an Eulerian multi-components model, and an Eulerian-Lagrangian model that we are developing. The relatively simple case of large amplitude motion of a bubble in a bubbly mixture is considered. The effect of the presence of the bubbles on the primary bubble dynamics, on the propagation of pressure waves in the medium, and on the distribution in space and time of the void fraction are investigated for various mixture properties. Our objective in this study is to evaluate the limits of validity of the application of simplified continuum media approaches to bubbly problems as well as to elucidate the role bubbles play at their microscopic level on the pressure and velocity fields. The selected problem enables us also to examine the accuracy of the Eulerian-Lagrangian approach that we are developing through comparison to parallel on-going experimental work.

2. EULERIAN-LAGRANGIAN MODEL

2.1. Viscous Flow Solver with Density Variations

The unsteady Navier-Stokes equations for a liquid-gas mixture written in non-dimensional form are as follows:
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\[
\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m u_i}{\partial x_i} = 0, \tag{1}
\]

\[
\frac{\partial \rho_m u_i}{\partial t} + \frac{\partial \rho_m u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ v_m \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \rho_m \delta_{ij} \frac{1}{F_r^2}, \tag{2}
\]

where \( \rho_m \) is the mixture density, \( u \) its velocity, \( p \) its pressure, \( v_m \) its kinematic viscosity and \( F_r = \frac{U_{\infty}}{\sqrt{gL}} \) is the Froude number expressing the relative amplitude of inertial and gravity pressure forces. \( U_{\infty} \) and \( L \) are respectively the problem characteristic velocity and length, and \( g \) is the acceleration of gravity. The mixture density and viscosity can be defined as,

\[
\rho_m = (1-\alpha) \rho_l + \alpha \rho_g, \quad v_m = (1-\alpha) v_l + \alpha v_g, \tag{3}
\]

where the subscripts ‘l’ and ‘g’ represent liquid and gas properties respectively while \( \alpha \) is the gas volume fraction.

Equations (1) and (2) are solved using our flow solver 3DYNAFS-Vis© based on the artificial-compressibility method [12], where a time derivative of the pressure is added to (1) as:

\[
\frac{1}{\beta_c} \frac{\partial p}{\partial \tau} + \frac{\partial \rho_m u_i}{\partial x_i} = -\frac{\partial \rho_m}{\partial t}, \tag{4}
\]

where \( \beta_c \) is an artificial compressibility factor and \( \tau \) is a pseudo-time. Equations (2) and (4) form a hyperbolic system of equations and are solved using a time marching scheme in pseudo-time. The time variation of the density is enforced as source terms in (2) and (4). To obtain a time-dependent solution, a Newton iterative procedure, where pseudo-time stepping in \( \tau \) is used at each physical time step \( t \), in order to satisfy the continuity equation.

The numerical scheme uses a finite volume formulation. A first-order Euler implicit difference scheme is applied to the time derivatives. The spatial differencing of the convective terms uses a flux-
difference splitting scheme based on Roe’s method [13] and a van Leer’s MUSCL method [14] for obtaining the first or third-order fluxes. A second-order central differencing is used for the viscous terms. The flux Jacobians required in an implicit scheme are obtained numerically. The resulting system of algebraic equations is solved using a discretized Newton Relaxation method in which symmetric block Gauss-Seidel sub-iterations are performed before the solution is updated at each Newton iteration. The 3DYNASFS-VIS© solver has been used to study a range of problems in the past and compared favorably with available experimental data [15,16,17].

2.2. Lagrangian Bubble Dynamics Model

Bubble volume variations are obtained using the Surface Averaged Pressure (SAP) model [17,18]. The equivalent bubble radius, \( R(t) \), is obtained using a modified Rayleigh-Plesset-Keller-Herring equation [19] which accounts for the surrounding medium compressibility and non-uniform pressure field:

\[
\left(1 - \frac{\dot{R}}{c_m}\right) \ddot{R} + \frac{3}{2} \left(1 - \frac{\dot{R}}{3c_m}\right) \dot{R} = \frac{\left(\mathbf{u}_{\text{enc}} - \mathbf{u}_b\right)^2}{4} + \frac{1}{\rho_m} \left(1 + \frac{\dot{R}}{c_m} + \frac{R}{c_m} \frac{d}{dt}\right) \left[p_v + \rho_{g0} \left(R_0/R\right)^k - p_{\text{enc}} - \frac{2\gamma}{R} - 4\mu_m \frac{\dot{R}}{R}\right].
\]  

(5)

c_m is the sound speed in the medium where the bubble is located, \( \mu_m \) is the medium viscosity, \( p_v \) is the vapor pressure, \( \rho_{g0} \) is the initial gas pressure, \( k \) is the gas polytropic constant, and \( \gamma \) is the surface tension. \( p_{\text{enc}} \) and \( \mathbf{u}_{\text{enc}} \) are the medium pressures and velocities averaged over the bubble surface and \( \mathbf{u}_b \) is the bubble center velocity. Their introduction in Equation (5) is to account for slip between the bubble and the host medium, and for non-uniform pressures along the bubble surface. The use of \( p_{\text{enc}} \) results in a major improvement over classical models, which use the pressure at the bubble center [17,18]. The bubble trajectory is obtained using the following motion equation [20]:

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\[
\rho_b V_b \frac{du_b}{dt} = -V_b \nabla p - \frac{1}{2} \rho_m V_b \left( \frac{Du_{enc}}{Dt} - \frac{du_b}{dt} \right) + (\rho_b - \rho_m) 2V_b g + \frac{1}{2} \rho_m A_b C_D \left( u_{enc} - u_b \right) \left| u_{enc} - u_b \right| + \\
R^2 C_L \sqrt{\rho_m \mu_m} \left( \frac{u_{enc} - u_b}{\sqrt{\left| \mathbf{O} \right|}} \times \mathbf{O} \right) + \rho_m V_b \frac{3}{R} \left( u_{enc} - u_b \right) \hat{R},
\]

The last term in the right hand side of (6) is a classical force due to the bubble volume variations. Since for the bubble $\rho_b \ll \rho_m$ and $\rho_m \left( \frac{Du_{enc}}{Dt} \right) = -\nabla p$ it becomes:

\[
\frac{du_b}{dt} = -\frac{3}{\rho_m} \nabla p - 2g + \frac{3}{4} \frac{C_D}{R} \left( u_{enc} - u_b \right) \left| u_{enc} - u_b \right| + \frac{3}{2\pi} \frac{C_L}{R} \sqrt{\frac{\mu_m}{\rho_m}} \left( \frac{u_{enc} - u_b}{\sqrt{\left| \mathbf{O} \right|}} \times \mathbf{O} \right) + \frac{3}{R} \left( u_{enc} - u_b \right) \hat{R},
\]

where $\rho_b$ is density of gas in the bubble, $A_b$ its projected area, $V_b$ its volume, $C_L$ is the lift coefficient, $\omega$ is the local vorticity, and $C_D$ is the drag coefficient given by, [15,21]:

\[
C_D = \frac{24}{R_{eb}} \left( 1 + 0.197 R_{eb}^{0.63} + 2.6 \times 10^{-4} R_{eb}^{1.38} \right), \quad R_{eb} = \frac{2 \rho R \left| u_{enc} - u_b \right|}{\mu}.
\]

2.3. Eulerian-Lagrangian Coupling

The coupling between the Eulerian (two-phase medium) and Lagrangian (bubble dynamics) approaches is as follows. The space and time dependent densities and viscosities of the two-phase medium are derived from the bubble sizes and locations obtained from the solution of the Lagrangian model. The bubbles’ volume changes, motion, and concentration are determined by the pressure and velocity fields computed by the Eulerian continuum two-phase medium solver.

For the physical problem studied here (Fig. 1), only representative bubbles are distributed in the radial direction, i.e. not each single bubble in the field is computed. Each of the representative bubbles represents $m_f$ bubbles located between it and its neighbors, i.e. between radial locations $[d_i, d_{i+1}]$. The void fraction, $\alpha$, is then given by:

\[
\alpha = \sum \frac{4}{3} \pi m_f R_i^3 \sqrt{\frac{4}{3} \pi \left( d_i^3 - d_{i-1}^3 \right)}.
\]
In this discretized approach, at each time step the void fraction, \( \alpha \), is calculated at the bubble location and the radial variations of \( \alpha \) are determined by linear interpolation between locations.

![Diagram showing representative bubbles and computed void fraction](image)

**Fig. 1**: Schematic showing representative bubbles used in the computations, the computed void fraction, and its interpolation onto the computational grid.

### 3. EULERIAN MULTICOMPONENT COMPRESSIBLE SOLVER

Compressible mixture flow calculations were carried out using the Eulerian finite difference solver, GEMINI, developed by the Naval Surface Warfare Center, Indian Head [22,23]. GEMINI is a multi-component compressible code, which uses a high order Godunov scheme. It employs the Riemann problem to construct a local flow solution between adjacent cells. The numerical method tracks each material and is based on single material higher order MUSCL scheme. To improve efficiency, an approximate Riemann problem solution replaces the full problem. The MUSCL scheme is augmented with a Lagrange re-map treatment of mixed cells. The code has been extensively validated against experiments [22,23].
4. CONTINUUM ANALYTICAL MODEL

The Gilmore equation [24]

\[
\left(1 - \frac{\dot{R}}{c_m}\right) R \dot{R} + \frac{3}{2} \left(1 - \frac{\dot{R}}{3c_m}\right) \dot{R}^2 = H \left(1 + \frac{\dot{R}}{c_m}\right) + \frac{R \dot{H}}{c_m} \left(1 - \frac{\dot{R}}{c_m}\right),
\]

(10)
governs spherical bubble dynamics in an infinite domain compressible medium of space variable density and sound speed. We will use it as a very simple reference solution. We describe it here succinctly, according to the derivations in [24]. In the above equation, \( H \) is the enthalpy difference between the bubble surface and infinity.

\[
H = \int_{p_v}^{\rho_{\text{super}}} \frac{dp}{\rho},
\]

(11)
and the pressure \( P_{\text{bubble}} \) can be expressed using a classical pressure balance equation at the gas-liquid interface.

\[
P_{\text{bubble}} = p_v + p_{g0} \left(\frac{R_0}{R}\right)^{3k} - 2\gamma \frac{4\mu \dot{R}}{R}.
\]

(12)
The bubble is supposed to contain gas of initial pressure \( p_{g0} \), which follows a polytropic law of compression constant, \( k \), and vapor pressure of constant pressure \( p_v \). \( \gamma \) is the surface tension parameter and \( \mu \) is the liquid viscosity. The last two terms in (12) express the normal stresses due to surface tension and viscosity. If an equation describing the pressure-density relationship for the medium is known, the above equations can be solved to obtain the variation of bubble radius with time.

4.1. Pure Water

For pure water, the pressure-density relation for isentropic compression can be given by the Tait equation [25]:

\[
\]
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\[ \frac{p + B}{P_\infty + B} = \left( \frac{\rho}{\rho_\infty} \right)^n, \]  \hspace{1cm} (13)

where \( \rho_\infty \) is a reference density corresponding to the pressure \( P_\infty \). \( B \) and \( n \) are constants \(( B \approx 3000 \text{ atm and } n \approx 7 \text{ for water})\). The sound speed in a liquid at pressure, \( p \), can then be expressed as:

\[ c_m^2 = \frac{dp}{d\rho} = \frac{n(p + B)}{\rho}, \]  \hspace{1cm} (14)

and the sound speed at the bubble surface can be expressed as:

\[ c_{\text{bubble}} = c_{\infty} \left( \frac{P_{\text{bubble}} + B}{P_\infty + B} \right)^{\frac{n-1}{2n}}, \]  \hspace{1cm} (15)

where \( c_{\infty} \) is the reference sound speed corresponding to the reference pressure \( P_\infty \).

Using (11) and (13), one obtains the following expression for \( H \):

\[ H_{\text{bubble}} = \frac{n(P_\infty + B)}{(n-1)\rho_\infty} \left[ \frac{P_{\text{bubble}} + B}{P_\infty + B} \right]^{\frac{n-1}{n}} - 1 \]. \hspace{1cm} (16)

Once \( c_{\text{bubble}} \) and \( H_{\text{bubble}} \) are known, the system of equations describing the wall-motion dynamics of a spherical bubble can be solved.

4.2. Bubbly Medium

In a dispersed (small \( \alpha \)) two-phase medium where the density of the gas phase is negligible relative to that of the liquid phase, \( \rho_l \), and where surface tension is ignored, [9], the sound speed can be expressed as follows:

\[ c_m^2 = \frac{dp}{d\rho_m} = \frac{p}{\rho_l} \left[ 1 + \frac{\alpha}{1-\alpha} \right]^2 \frac{1}{k \frac{\alpha}{1-\alpha} + \frac{p}{\rho_l c_i^2}}, \]  \hspace{1cm} (17)
where \( c_i \) is the sound speed in the pure liquid and \( \rho_m = (1 - \alpha) \rho_i \), is the density of the two-phase medium.

Ignoring the compressibility of the pure liquid relative to that of the bubbly medium and expanding Equation (17) we obtain:

\[
-\rho_i \left[ \frac{1}{k} \frac{\alpha}{1 - \alpha} + \frac{p}{\rho_i c_i^2} \right] dp = \rho \left[ 1 + \frac{\alpha}{1 - \alpha} \right] \alpha \, d\alpha,
\]

This was integrated in [9] between reference state 0 and present state to obtain an equation of state of the bubbly medium relating pressure and density/void-fraction variations:

\[
\frac{\alpha}{1 - \alpha} = \left[ \frac{\alpha_0}{1 - \alpha_0} + \frac{k}{k + 1} \frac{p_0}{\rho_i c_i^2} \right] \left( \frac{p_0}{p} \right)^{1/k} - \frac{k}{k + 1} \frac{p}{\rho_i c_i^2},
\]

Equations (19) and (16) can be used to obtain \( H_{\text{bubble}} \):

\[
H_{\text{bubble}} = \frac{P_{\text{bubble}} - P_{\infty}}{\rho_i} + (A + B) \frac{k}{\rho_i (k - 1)} \rho_0 \left( \frac{P_{\text{bubble}}}{\rho_i} - \frac{P_{\text{bubble}}^{k-1}}{\rho_i^{k-1}} \right) - \frac{B}{2 \rho_0 \rho_i} \left( P_{\text{bubble}}^{2/k} - P_{\infty}^{2/k} \right),
\]

where \( A = \alpha_0/(1 - \alpha_0) \) and \( B = k \rho_0 / (k + 1) \rho_i c_i^2 \).

![Schematic showing an oscillating bubble in a bubbly mixture.](image)
5. PROBLEM SETUP

5.1. Eulerian-Lagrangian Setup

A bubble of initial radius $R_0$ is placed in a bubbly mixture composed of uniform size bubbles and with a constant void fraction distribution in space (Fig. 2). This setup resembles previously studied spherical bubble clouds [9, 26, 27, 28], however, one notable difference is that here the primary bubble acts as the motion/pressure source or driver while responding to the behavior of the surrounding bubbles. This also corresponds to our parallel on-going experimental tests of spark generated bubbles in bubbly media [29].

Even though this is a 1D problem, the full 3D 3DYNAFS© Eulerian-Lagrangian code is used for development and validation purposes. An O-O type single block grid is used for the Eulerian problem (Fig. 3). For most computations shown here, the bubble surface has $41 \times 21$ nodes, while the flow field is resolved using 25 nodes in the radial direction, clustered near the bubble surface. General boundary conditions (kinematic and dynamic) are imposed at the free surface [30]. The kinematic condition is that a particle on the surface remains on the surface; for a surface of equation $F(x,t) = 0$, this can be expressed as $DF / Dt = 0$. The dynamic condition imposes zero shear stress and balance of normal stresses at the interface. Hodges et al. [31] derived a dynamic curvilinear coordinate boundary condition by requiring the grid to be normal to the boundary. We apply this for non-inertial curvilinear $(\xi, \eta, \zeta)$ coordinates and further assume that $\partial W / \partial \xi$ and $\partial W / \partial \eta$ are small to obtain in non-dimensional format:

$$\frac{\partial U}{\partial \zeta} \bigg|_{\zeta=0} = 0, \quad (21)$$

$$\frac{\partial V}{\partial \zeta} \bigg|_{\zeta=0} = 0, \quad (22)$$
where \((U, V, W)\) are the contravariant velocity components in the curvilinear coordinates, \(C_b\) is the bubble curvature, and

\[
W_{eb} = \rho u_{\text{char}}^2 L / \gamma .
\]  

\((24)\)

\(u_{\text{char}}\) and \(L\) are a characteristic velocity and length, respectively. The term \(p_{av}\) in Equation (23) is the non-dimensional pressure difference between the pressure inside the bubble and the reference pressure, \(P_\infty\),

\[
p_{av} = \frac{1}{\rho u_{\text{char}}^2} \left[ p_0 \left( \frac{R_0}{R} \right)^{3k} + p_i - P_\infty \right].
\]  

\((25)\)

At the far field discretized boundary (selected at a radial distance of 100 \(R_0\)) an extrapolation boundary condition is used.

**Fig. 3**: Grid used for the viscous Eulerian-Lagrangian 3DynaFS-VIS© calculations.

As the bubble oscillates the grid is modified using a combination of algebraic and elliptic grid generation techniques [32]. This scheme is appropriate because the algebraic technique allows
A grid refinement study was conducted to determine the suitability of the grid for the problem at hand (see Fig. 4). Since the solution varies only in the radial direction, the grid was refined by doubling the grid in this direction till both changes in the period and maximum radius change were less than 0.6%. The numerical uncertainties for the grid used in most of the calculations below were estimated by comparing the results with those of the finest grid and were estimated to be 0.1% for the maximum bubble radius and around 1.0% for the bubble period. The results presented in the paper are for the base grid described earlier.

Fig. 4: Grid convergence study for the viscous Eulerian-Lagrangian 3DynaFS-Vis© calculations showing errors in \(R_{max}\) and bubble period vs. the number of grids in the radial direction.

5.2. Compressible Multi-component Code Setup

For the compressible computations using GEMINI, the bubbly medium is modeled as a homogeneous medium composed of air and water. The primary bubble surface is followed using an interface capturing scheme in the spherical coordinates. Non-reflecting boundary conditions are used in the far field selected at a radial distance of 2,000 \(R_0\). 648 grid points were used in the radial direction, after this was found sufficient to capture the pressure characteristics for the current setup.
Following the procedure mentioned in the previous section, numerical uncertainty for the compressible multi-component solver was estimated from a grid convergence study to be 0.002% for the maximum bubble radius and 0.015% for the bubble period respectively.

6. COMPARISON OF THE RESULTS

6.1. Example Test Case

Let us consider a primary bubble with $R_0 = 5$ mm, $P_{g0} = 2$ atm in an ambient pressure of 1 atm. The medium surrounding this bubble is formed of water and bubbles of initial radii, $r_0 = 1$ mm. We consider the behavior of this bubbly medium and primary bubble for void fractions ranging from 0 to 1%.

![Graph showing primary bubble radius versus time for different initial void fractions obtained with the Eulerian-Lagrangian calculations. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.](image)

Fig. 5: Primary bubble radius versus time for different initial void fractions obtained with the Eulerian-Lagrangian calculations. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
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Fig. 6: Primary bubble radius versus time for different void fractions obtained with the multi-component compressible approach calculations. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.

Fig. 7: Primary bubble radius versus time for different void fractions calculated using the Gilmore analytical model. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
6.2. Primary Bubble Radius Time Variations in Different Void Fraction Media

Figure 5 compares the primary bubble radius versus time for different void fractions from $\alpha_0 = 0$ to 1% obtained with the Eulerian-Lagrangian approach, while Fig. 6 and Fig. 7 show the same obtained with the multi-component compressible approach and the Gilmore analytical approach. In all three figures the Rayleigh-Plesset [19] solution is also shown.

All three models clearly show that the amplitude of the oscillations of the primary bubble is very much affected by the presence of the surrounding bubbly medium. This results in significant damping of these oscillations (smaller maximum radius and larger minimum radius) and modification of the bubble period, and this effect increases with the void fraction. The maximum radius reduction and the phase shift are directly related to the modification of the compressibility of the medium surrounding the bubble due to the presence of other bubbles. The fact that it depends so strongly and directly on $\alpha$ is an indication of this dependence. Viscous damping may play a role but this would be over many cycles and this effect would not be so much affected by changes in $\alpha$. This damping also appears stronger when bubble dynamics is accounted for, resulting in reduction of the peak-to-peak oscillations of 15%, while the other methods give 10% reduction. Similarly, a reduction of the period of the primary bubble appears to be captured only when medium bubble dynamics is taken into account. This reduction was also observed experimentally [29] as seen in Fig. 8 (a). This effect is due to exchange of energy between the main bubble and the surrounding small bubbles. Energy is absorbed by the surrounding bubbles and is then reradiated as pressure fluctuations which in turn affect the primary bubble. These are the primary effects our new Eulerian-Lagrangian code aims at capturing.
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Fig. 8: (a) Evolution of the normalized primary bubble radius vs. normalized time – Experiments from [29] versus Gilmore analytical model and (b) comparison of void fraction computed from experiments and analytical model, [29].

Fig. 9: Comparison of the ratio of maximum radius, $R_{\text{max}}$, achieved in the two-phase medium to its value in water only, for the three numerical models. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{\text{amb}} = 1$ atm.
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Fig. 10: Comparison of the bubble period achieved in the two-phase medium to its value in water only, for the three numerical models. $R_0 = 5 \text{ mm}, P_{g0} = 2 \text{ atm}, r_0 = 1\text{ mm}, P_{amb}= 1\text{ atm}$.

The presence of bubbles in the surrounding medium also has an effect on the primary bubble maximum bubble radius, as shown in Fig. 9. All three approaches show that as the void fraction increases the maximum bubble radius decreases. The computed effect is also stronger with the Eulerian-Lagrangian approach, which shows a 5% reduction of $R_{\text{max}}$ at $\alpha_0=1\%$, while the analytical approach shows only 3% reduction.

A similar comparison for the period of the bubble, normalized by the bubble period achieved in a pure liquid (at $\alpha_0 = 0$) is shown in Fig. 10. There is a significant difference in bubble period between the three approaches. The analytical model shows a marginal change in the bubble period while the Eulerian-Lagrangian approach shows that the bubble period decreases by nearly 12% at $\alpha_0 = 1\%$ and the multi-component compressible approach shows a 3% decrease.
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Fig. 11: Variation of the pressure at $r = 7$ mm for different void fractions for the Eulerian-Lagrangian approach. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.

Fig. 12: Variation of the pressure at $r = 7$ mm for different void fractions for the multi-component compressible approach calculations. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
Fig. 13: Variation of the pressure at $r = 7$ mm for different void fractions calculated using Gilmore’s model. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.

6.3. Field Pressure Variations

Figure 11, Fig. 12 and Fig. 13 show the pressure time variations for different $\alpha_0$’s at $r = 7$ mm for each of the three models. It is clearly seen that the bubbly mixture attenuate the pressure, with the attenuation increasing as the void fraction increases. The Eulerian-Lagrangian approach captures higher frequency pressure oscillations, more visible with the increase in the void fraction, as observed in Fig. 11 near the minimum pressure. This is attributed to the accounting for the behavior of the discrete field bubbles which reradiate the energy they absorb from the main bubble at their own frequency and add their local pressure emissions to the average medium pressures (this is further illustrated in Fig. 16 later). The two other homogeneous mixture solvers, by definition, cannot capture these modulations. The initial pressure jump in the multi-component compressible approach (Fig. 12) is due to a wave generated by the imposed initial pressure discontinuity at the bubble wall. For the
Eulerian-Lagrangian approach a sharp drop in pressure is seen initially and is due to strong interaction between the field and the bubbles close to \(r=7\ mm\).

**Fig. 14:** Comparison of the time variation of the pressure at \(r = 7\ mm\) for \(\alpha_0 = 0\%\) for the three approaches. \(R_0 = 5\ mm, P_{g0} = 2\ atm, r_0 = 1\ mm, P_{amb} = 1\ atm\).

**Fig. 15:** Comparison of the time variation of the pressure at \(r = 7\ mm\) for \(\alpha_0 = 1\%\) for the three approaches. \(R_0 = 5\ mm, P_{g0} = 2\ atm, r_0 = 1\ mm, P_{amb} = 1\ atm\).
The pressure time variations are compared in Fig. 14 and Fig. 15 for $\alpha_0=0$ and $\alpha_0=1\%$ respectively for all three approaches. In the pure liquid the peak pressures when the bubble is at its minimum radius have the same magnitude as the imposed initial bubble pressure and are comparable to the pressure obtained using the Rayleigh-Plesset solution. This is not the case for the minimum pressures achieved when the bubble radius is at its maximum. This is a non-linear effect connected to the different $R_{max}$ values obtained by each of the four methods and requires future investigation. Since the two analytical methods give close results, grid fineness could be responsible for the observed differences in the numerical methods and will be studied carefully.

For $\alpha_0 = 1\%$ all three approaches indicate that the field pressure is significantly attenuated and that the primary bubble loses energy to activate the bubbly medium. There are differences between the results of the three approaches, but these differences are of the same order as those for the pure liquid.

6.4. Void Fraction Field Variations

The differences between the three approaches are further examined by analyzing the time variation of the void fractions and the pressures at different locations, when the initial void fraction is 0.1%. Figure 16 shows these time variations at three radial locations ($r = 7, 15$ and $400 \text{ mm}$) for the Eulerian-Lagrangian model. For the point closest to the bubble surface, the pressures and void fractions show the largest variations, strongly illustrating the volume oscillations of the field bubbles. Further away from the bubble, the void fraction and pressure oscillations are still present but are weaker as the distance increases. These trends were also observed in our on-going experiments with spark bubbles in bubbly media [29] but cannot be captured by the other two approaches. This can be easily explained by the interaction between the main bubble and the closest neighboring bubbles. This effect will obviously depend on the relative sizes of the main and satellite bubbles and the study here just gives an indication of this interaction. The study needs to be expanded to cover a broader
range of conditions and a polydisperse distribution of bubble sizes. As we move away from the primary bubble, the pressure field it generates is attenuated and its effect on the satellite bubbles is reduced resulting in a weaker interaction. Figure 8(b) shows the oscillations in the void fraction at a field point captured in the experiments [29] which confirms this phenomenon. The multi-component compressible approach and the Gilmore approach, both using the homogeneous mixture assumption with no particular two-phase medium bubble behavior, show instead smooth pressure and void fraction variations (Fig. 17 and Fig. 18). These capture average type quantities over space and time. All three models predict similar maximum void fractions: 0.12% for the Eulerian-Lagrangian solver, 0.14 % for the multi-component compressible approach, and 0.16 % for the Gilmore model.

![Graph](image)

**Fig. 16: Variation of pressure and void fraction in the field with $\alpha_0 = 0.1\%$ for the Eulerian-Lagrangian calculations, at $r = 7, 15$ and 400 mm. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.**
Fig. 17: Variation of pressure and void fraction in the field with $\alpha_0 = 0.1\%$ for the multi-component compressible approach calculations, at $r = 7, 15$ and $400$ mm.

Fig. 18: Variation of pressure and void fraction in the field with $\alpha_0 = 0.1\%$ calculated using the analytical Gilmore model, at $r = 7, 15$ and $400$ mm. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
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Fig. 19: Comparison of the pressure in the field for the three approaches at different phases of the bubble period (0.25T, 0.5T, and 0.75T) for $\alpha = 1\%$. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
Fig. 20: Comparison of the void fraction in the field for the three approaches at different phases of the bubble period (0.25T, 0.5T, and 0.75T) for $\alpha_0 = 1\%$, $R_0 = 5 \text{ mm}$, $P_{g0} = 2 \text{ atm}$, $r_0 = 1 \text{ mm}$, $P_{\text{amb}} = 1 \text{ atm}$. 
Figure 19 compares the pressure distributions in space for $\alpha_0 = 1\%$ at different time instances ($0.25T$, $0.5T$, and $0.75T$) for the Eulerian-Lagrangian approach during the bubble period, $T$. At $t=0.25T$, a high pressure wave confined to the bubble surface neighborhood is seen. All three approaches show similar profiles at this time with the discrete bubble showing a more symmetric pressure peak structure. During the later stages the two homogeneous mixture models give similar distributions, while the Eulerian-Lagrangian discrete bubble approach continues to indicate a smoother symmetric pressure hump.

![Figure 21: Snapshots from a high speed movie of primary bubble growth and collapse in water with bubble injection, reproduced with permission from [29].](image)

The corresponding void fraction distributions in the field for all three approaches are compared in Fig. 20 at $0.25T$, $0.5T$, and $0.75T$. Again, the two homogeneous mixture approaches match well, while the discrete bubble model, most probably due to its accounting for the bubble dynamics and for a slip velocity between the bubbles and the liquid, indicates a strong decay in void fraction near the primary bubble wall as the field bubbles move away during the bubble dynamics. Indeed, consideration of the dynamics of single satellite bubbles close to the primary bubble indicates that the small bubbles move much faster than the liquid during the growth phase of the primary bubble (Bjerkness effect) and thus they move away fast from the interface leaving a region of pure liquid. The starving of this region of bubbles during the primary bubble growth is naturally completely missed by the homogenous models. Figure 21 shows the primary bubble dynamics and the effect of its growth on the surrounding bubbles as seen in experiments [29]. Due to high pressure gradients near the bubble surface the surrounding
bubbles are pushed away initially, as seen in numerical results and then are pulled into a layer around the primary bubble.

6.5. Mixture Bubble Dynamics – One and Two-way Coupling

Figure 22 compares the changes in the mixture bubble radii over time computed using two different discrete models. Two-way coupling computations are obtained with the Eulerian-Lagrangian approach and are compared with bubble computations using the solution from the GEMINI multi-component compressible approach. Bubbles at three different locations ($r = 7, 15$ and $400 \text{ mm}$) are examined. Results indicate that the one-way bubbles oscillations have higher magnitudes than the two-way coupled bubbles. This is attributed to the fact that, in the coupled model, changes in the void fraction are fed-back and hence the bubbles in the medium “feel” a lower pressure than the one-way coupled cases. As a result of these reduced encountered pressures, smaller radii oscillations are observed here for the coupled cases. These results will strongly depend on the relative sizes between the bubbles and a more general study is needed to cover the range of potential configurations. We start some of this investigation in the following paragraph.

6.6. Effect of the Mixture Bubble Size

We have imputed in the previous sections time and space fluctuations of pressures and velocities in the discrete model to the presence of the bubbles in the two-phase medium. To further illustrate this, we consider the effect of changing the size of the field bubbles on the results, while conserving the initial void fraction. Figure 23 shows the pressure and the void fraction variations in the field when $r_0 = 2 \text{ mm}$. By comparing this to the baseline case in Fig. 16 one can observe that with the twice larger satellite bubbles, the period of oscillations has doubled and the amplitude of the void fraction almost tripled.
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**Fig. 22:** Comparison of the bubble radius in the field at $r = 7$, $15$ and $400$ mm for two-way and one-way coupled approaches. $R_0 = 5$ mm, $P_{g0} = 2$ atm, $r_0 = 1$ mm, $P_{amb} = 1$ atm.
Fig. 23: Variation of pressure and void fraction in the field for $\alpha_0 = 0.1\%$ and $r_0 = 2\text{mm}$, for Eulerian-Lagrangian calculations. $R_0 = 5 \text{ mm}$, $P_{g0} = 2 \text{ atm}$, $P_{amb} = 1 \text{ atm}$.

Fig. 24: Comparison of the time variation of the void fraction at $r = 7 \text{ mm}$ for three initial field bubble sizes ($r_0 = 0.5$, 1, and 2mm), calculated using Eulerian-Lagrangian approach. $\alpha_0 = 0.1\%$, $R_0 = 5 \text{ mm}$, $P_{g0} = 2 \text{ atm}$, $P_{amb} = 1 \text{ atm}$.
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**Fig. 25:** Comparison of the time variation of the pressure at \( r = 7 \text{ mm} \) for for three initial field bubble sizes \((r_0 = 0.5, 1, \text{ and } 2\text{mm})\), calculated using Eulerian-Lagrangian approach. \( \alpha_0 = 0.1\% \), \( R_0 = 5 \text{ mm} \), \( P_{g0} = 2 \text{ atm} \), \( r_0 = 1 \text{mm} \), \( P_{amb} = 1 \text{atm} \).

The effect of the size of the field bubbles on the void fractions versus time are compared in Fig. 24 for a point close to the bubble surface \((r=7\text{mm})\). By increasing the field bubble sizes we are bringing their natural frequency of oscillations closer to that of the primary bubble [33]. The figures illustrate the natural trend of larger bubbles to oscillate with larger amplitudes and periods. In addition, as the bubble sizes approach that of the primary bubble size, the tendency for disappearance of secondary oscillations and occurrence of resonance increases.

The variations in the field pressures corresponding to Fig. 24 are shown in Fig. 25, and reflect the oscillations due to the local changes in void fraction. Higher frequencies of the oscillations of the smaller bubbles cause perturbations in the pressure which last throughout the period of the primary bubble. These oscillations are significantly damped with larger bubbles. The average pressures have however all a similar behavior, with larger bubbles having a stronger damping effect on the pressure wave oscillations.
7. CONCLUSIONS

We have examined the validity of using three distinct approaches: two compressible continuum homogeneous models and a coupled Eulerian-Lagrangian variable density discrete bubble model, to study pressure wave propagation due to large bubble oscillations in a bubbly medium. While the continuum models capture the average low-frequency behaviors, the microscale response of the discrete bubbles is only captured by the Eulerian viscous solver coupled with a Lagrangian discrete bubble dynamics solver. The study has shown that the surrounding bubbly medium, with bubble sizes smaller than the primary bubble, absorb the energy radiated from the primary bubble thereby damping the resulting bubble oscillations. All models show the important result of reduction respectively of both the maximum bubble radius achieved and the bubble period when the bubbly medium is taken into account. The Eulerian-Lagrangian model shows a larger effect than the two continuum models. Also all three models capture the attenuation of the pressure in the field occurring with increasing void fractions. The Eulerian-Lagrangian model captures in addition higher frequency pressure oscillations due to the individual small bubbles dynamics.

The interaction between the primary bubble and the bubbly medium is a highly nonlinear phenomenon with the satellite bubbles oscillating at their own frequency. This is captured only by the coupled multiscale model. Higher frequency oscillations in the void fraction are seen near the primary bubble while further away the interactions become weaker due to lower excitation. The coupled model also captures the starvation of the region surrounding the primary bubble during the growth phase due to the higher slip velocity between the satellite bubbles and liquid in that region. These observations qualitatively follow parallel experiments showing spatial fluctuations of both pressure and void fractions as well as the starving of bubbles. Although the two continuum approaches match each other well and do capture the global trend of the primary bubble, they are unable to reproduce these local fluctuations suggesting that are not entirely suitable for such problems. These fluctuations
are found to be a function of the field bubble sizes, with smaller bubbles reflecting higher frequency oscillations.

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NOMENCLATURE

$A, B$ Constants
$A_r$ Area
$C$ Sound speed on bubble surface
$C_b$ Curvature of the bubble interface
$C_D$ Drag coefficient
$C_L$ Lift coefficient
$F_r$ Froude number
$H$ Enthalpy difference
$L$ Characteristic Length
$P$ Pressure at bubble wall
$P_{amb}$ Ambient Pressure
$P_{\infty}$ Pressure at infinity
$r$ Radial location
$r_0$ Initial bubble size in bubbly medium
$R$ Bubble radius
$R_{max}$ Maximum radius of primary bubble
$R_e$ Reynolds number
$U, V, W$ Contravariant velocity components
$W_{eb}$ Weber number
$c$ Sound speed in the bubbly medium
$c_{\infty}$ Reference sound speed
$d$ Mid-distance between bubbles
$g$ Gravity
$k$ Polytropic gas constant
$m_f$ Bubble multiplication factor
$n$ Constant
$p$ Pressure
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\( p_v \)  
Vapor pressure

\( t \)  
Time

\( T \)  
Time period of the primary bubble

\( u \)  
Velocity

\( u_{\text{char}} \)  
Characteristic velocity

\( V \)  
Volume

\( x \)  
Directional vector

\( \alpha \)  
Void fraction

\( \beta_c \)  
Artificial compressibility factor

\( \gamma \)  
Surface tension of compressible medium

\( \gamma_b \)  
Surface tension at the bubble interface

\( \Omega \)  
Vorticity

\( \rho \)  
Density of the medium

\( \rho_\infty \)  
Reference density

\( \mu \)  
Absolute Viscosity

\( \nu \)  
Kinematic Viscosity

\( \tau \)  
Pseudo-time

\( \xi, \eta, \zeta \)  
Curvilinear coordinates

**Subscripts**

\( 0 \)  
Initial condition

\( b, \text{bubble} \)  
Bubble property

\( \text{enc} \)  
Average property on bubble surface

\( i, j, k \)  
Indices

\( g \)  
Gas property

\( l \)  
Liquid property

\( m \)  
Mixture property

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