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Cloud Cavitation: Theory

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ABSTRACT

The collapse of a bubble cloud, due to a change in the surrounding pressure, is considered, first, by using a single perturbation theory. The interaction of any individual cavity with the rest of the cloud is modelled using matched asymptotic expansions in powers of the ratio between the characteristic bubble radius and interdistance. Up to the third order, the problem is shown to be equivalent to the interaction of two cavities of different collapsing strengths. The numerical results obtained with a symmetrical repartition of bubbles on a spherical shell show that the influence of the other bubbles in the cloud on the collapse of a particular bubble is to reduce the driving pressure during most of the collapse time, thus delaying the implosion, and then to dramatically increase this pressure producing a violent end of the collapse. The pressure released is then orders of magnitude higher than with an isolated bubble. This pressure, which is being imposed on an area of the same size as the whole cloud, could explain the high erosion rates and bending of foil trailing edges. In the second part, a continuum medium approach of the cloud is considered in order to extend the validity of the preceding approach to higher void fraction and to enable to account for the compressibility of the bubbly medium.

I. INTRODUCTION

The design criteria for high-speed ship propellers involve trade-offs between efficiency and cavitation, and strength and vibration of the propeller. Operating in ship wakes at relatively low cavitation numbers, the propeller will, at least intermittently, cavitate, inducing erosion of the blades, loss of efficiency, noise, vibration, and occasionally structural failure of nearby plating. These harmful effects are mainly due to the collapse of unsteady cavities. These include individual bubbles as well as sheet cavities and "clouds" (Figure 1).

Adequate and increasingly sophisticated theories for individual bubble growth and collapse exist (see the reviews by Plesset and Prosperetti, 1977 and Hammitt, 1980). While the transition to sheet
cavity is not well-understood, a large number of experimental observations of sheet cavitation are available (Shen and Peterson, 1979, Bark and Barlekom, 1979), and a steady, then unsteady, theory for sheet cavitation was recently developed (Tulin, 1980 and Tulin and Hsu, 1980). Downstream of a "steady" sheet cavity a region of high population of tiny bubbles can be observed and is especially known to be associated with erosion. However, as concluded from observations by Tanabayachi and Chiba (1977), an unsteady sheet cavity is required for the formation of coherent clouds of very fine bubbles. These clouds are either detached from the frothy mixture at the trailing end of the unsteady sheet, or generated in a finite region of the liquid downstream of the unsteady sheet where significant fluctuating pressures exist.

As the pressures generated by single bubble collapse are not strong enough to explain the intense erosion in the subject region, and the high forces needed, for example, to bend the trailing edge, cloud cavitation has been held responsible since Van Manen's (1963) work. This is supported experimentally by a very close correlation between the dynamics of these clouds and the sharpest and highest pressure pulses detected on an oscillating hydrofoil (Bark and Barlekom, 1979). Similar phenomena have been observed with ultrasonic cavitation (Hanson and Mörch, 1980).

Apart from some information on the frequency of generation of cloud cavitation, the experimental observations and measurements are very qualitative and do not allow at the present time, any quantitative predictions. In addition, the lack of understanding of the dynamics of such cavities makes it impossible to explain any scaling effects and to correct for them. Theoretical and fundamental studies are thus needed as guidance for future design and experimentation.

To our knowledge, since the early work of Van Wijngaarden (1964) only a few publications by Mörch (1977, 1980, and 1982) and Hanson and Mörch (1980) have dealt theoretically with the problem of "collective bubbles collapse" or "cavity cluster collapse." However a large amount of literature has been devoted to the modeling of bubble-liquid mixture behavior, using either a continuum medium approach or a "two-fluid" approach (Zwick, 1959; Van Wijngaarden, 1972, 1976, 1980, 1982; Zuber, 1964; Ishii, 1975). In order to explain the phenomenon of propeller blades bent at the trailing edge, Van Wijngaarden (1964) considered the case of a uniform layer of cavities on a solid wall. He studied its unidimensional collective collapse when the surrounding fluid is suddenly exposed to a pressure increase. He derived the continuity and momentum equations for the layer, neglecting the convective and dissipative terms and assuming that the volume fraction of gas is small enough to authorize such approximations. However, he took into account the individual bubble radial motion and translation, neglecting viscous effects. Solving the derived system of equations, Van Wijngaarden found a considerable increase of the pressure along the wall due to collective effects.

Mörch (1977, 1980), concerned with ultrasonic cavitation fields, considered the collapse of a hemispherical "cluster" near a wall, which by symmetry, he extended to the case of a spherical cloud. He
characterized the cloud only by its radius and a uniform volume fraction, $\alpha$, constant in time, and developed the following model. A pressure rise in the liquid leads to the formation of a shock wave on the "cloud boundary". The shock moves toward the cloud center leaving no bubbles behind it and thus constitutes the cloud boundary at each time. The collapse time of a spherical cloud is found to be $\sqrt{\alpha}$ times the Rayleigh collapse time of a spherical bubble of the same initial radius. Although a very interesting approach, especially for the calculation of the collapse time, this model (like Rayleigh's model for spherical bubble collapse) is incapable, in its present state, of adequately calculating the pressure field. At the end of the collapse the cloud radius is zero and the velocities and pressures are infinite, since the model does not allow the bubbles to contain noncondensables. In addition, the main physical assumption (presence of a shock wave dividing the space in two regions one containing bubbles which do not sense the pressure variations until a later time stage, and another one where all bubbles have collapsed) is valid only for relatively high void fractions. The case of a spherical single cavity of the same size as the whole cloud is the perfect extreme example of the domain of validity of this approach. Hanson and Mørch (1980) and Mørch (1982) extended the same model to a cylindrical cloud and a layer of bubbles on a solid plate.

We present in this paper first a singular perturbation theory (Chahine, 1981) which will allow us to compute the pressure field and the cloud dynamics for the growth and collapse of a cloud composed of a finite number of bubbles. In the second part of the paper we will discuss a continuum medium approach for a bubble cloud collapse.

II. SINGULAR-PERTURBATION THEORY

The following approach is applicable to a cloud of bubbles of low void fraction. Provided that the characteristic size of a bubble in the cloud, $r_b$, is small compared to the characteristic distance between bubbles, $\ell_0$, we can assume in absence of initial relative velocity between the bubble and the surrounding fluid that each of the individual bubbles reacts, in first approximation, to the local pressure variations spherically as if isolated. To the following order of approximations, interactions between bubbles induce bubble motion and deformation and are taken into account. This approach is an extension of the earlier studies by Chahine and Bovis (1981) and Bovis and Chahine (1981) on the collapse of a bubble near a solid wall and a free surface, later presented more generally for nonspherical bubbles by Chahine (1982).

Since the problem possesses two different geometrical scales, $\ell_0$ and $r_b$, we can consider two subproblems: one concerned with the macroscale and the other one with the microscale. The "outer problem" is that considered when the reference length is set to be $\ell_0$. This problem is concerned with the macrobehavior of the cloud, and the bubbles appear in it only as singularities. The "inner problem" is
that considered when the lengths are normalized by \( r_b \) and its solution applies to the vicinity of the considered individual bubble of center \( B^i \). The presence of the other bubbles, all located at infinity in the "inner problem", is sensed only by means of the matching condition with the "outer problem". That is to say, physically the boundary conditions at infinity for the "inner problem" are obtained, at each order of approximation, by the asymptotic behavior of the outer solution in the vicinity of \( B^i \). Mathematically, one has to match term by term the inner expansion of the outer solution with outer expansion of the inner solution, using the same asymptotic sequence in the two expansions.

A. Bubble Radius Variations

The determination of the flow field and the dynamics of any of the individual bubbles, \( B^i \), is accessible once the boundary conditions at infinity in the corresponding "inner region" are known. Here we imposed the restrictive assumption that the void fraction is low enough so that the information about the variation of the ambient pressure around the cloud, \( \bar{P}_\infty(t) \), is transmitted to the microscale in a time scale much shorter than the bubble collapse time. Therefore, in the absence of a slip velocity between the considered bubble and the surrounding fluid and when interactions are neglected, the only boundary condition at infinity is the imposed pressure variation \( \bar{P}_\infty(t) \). The "inner problem" is therefore spherically symmetrical and its solution is given by the well-known Rayleigh-Plesset equation. With the assumption that the liquid is inviscid and incompressible this equation can be written as follows:

\[
\frac{a^2}{a_0^2} + \frac{3}{2} \frac{\dot{a}}{a_0} = -\bar{P}_\infty(t) + \bar{P}_g(a_0^{-3k} - 1) + W_e^{-1}(1 - a_0^{-1}) \tag{1}
\]

In this equation, where the superscript \( i \) is omitted for convenience, \( a^2(t) \) is the radius of the bubble \( B^i \) normalized by \( r_b \). The times are normalized by the Rayleigh time based on \( r_b \) and \( \Delta P = \bar{P}_o - \bar{P}_v \). All pressures are normalized by \( \Delta P \) where \( \bar{P}_o \) is the initial pressure, and \( \bar{P}_v \) the vapor pressure. \( W_e \) is the Weber number and \( \bar{P}_g \) the initial normalized gas pressure in the bubble. The noncondensable gas pressure inside the bubble, \( \bar{P}_g \), is assumed to have a polytropic behavior, \( P_g a^{-3k} = \text{cte} \).

When interactions cannot be neglected, still assuming that an "inner region" enclosing the bubble \( B^i \) can be defined, the boundary conditions at infinity can be much more complex than in the preceding paragraph. First, as we will see in paragraph 3, the macroscale pressure in the cloud at \( B^i \), \( P(B^i, t) \), can be very different from the imposed far field pressure \( \bar{P}_\infty(t) \) and depends indeed on the bubble location in the cloud. Second, a relative velocity between the bubble and the surrounding fluid, \( \dot{U}(B^i, r, t) \) can exist causing the bubble to be nonspherical. Both \( P \) and \( \dot{U} \) can be determined only by solving the equations of motion of the two-phase medium as presented in paragraph 3. Here we will limit ourselves to a small perturbation theory whose interest will be to give the behavior of the solution when the perturbation grows continuously. In that case \( P(B^i, t) \), which is the

* underlined quantities are vectors
driving pressure for the collapse of the bubble \( B^i \), is only a perturbation of the imposed far field pressure, \( P_\infty (t) \), and \( U(B^i, r, t) \) is a perturbation of the spherical velocity due to the bubble volume variation.

If we assume that the liquid flow is irrotational, we can define a velocity potential for the macroscale ("outer problem"), \( \phi(B^i, t) \), and a velocity potential for the microscale ("inner problem"), \( \phi_1(B^i, r, t) \) both satisfying the Laplace equation. The matching condition between these two potentials expresses the at-infinity conditions for \( \phi_1 \), and replaces the conditions on \( P(B^i, t) \) and \( U(B^i, r, t) \). Using the results obtained with the interaction of two bubbles and the property of addition of potential flows, this condition can be written:

\[
\lim_{r \to \infty} \phi_1(B^i, r, t) = \sum_{j=1}^{N} \left( \frac{\lambda^j_0}{\lambda^j_0} \right) \left( \varepsilon q^j_0 + \varepsilon^2 q^j_1 + \varepsilon^3 q^j_2 + \ldots \right) +
\]

\[
+ \left( \frac{\lambda^j_0}{\lambda^j_0} \right)^2 \left( \varepsilon^2 q^j_1 + \varepsilon^3 q^j_1 + \ldots \right) r \cos \theta^j_1 +
\]

\[
+ \left( \frac{\lambda^j_0}{\lambda^j_0} \right)^3 \left( \varepsilon^3 q^j_1 + \ldots \right) r^2 P_2(\cos \theta^j_1) + \ldots \right] \quad (2)
\]

where the superscript \( (j) \) denotes quantities corresponding to the other bubbles, \( B^j \). \( \lambda^j_0 \) is the initial distance between the bubble centers \( B^i \) and \( B^j \). \( \theta^j_1 \) is the angle \( MB^iB^j \) and \( r \) the distance \( B^iM \), where \( M \) is a field point in the fluid (see Figure 2). \( P_2(\cos \theta) \) is the Legendre polynomial of order 2 and argument \( \cos \theta \). \( q^j_n \) is the correction of order \( \varepsilon^n \) of the strength, \( q^j_0 = a^j_0 \left( a^j_0 \right)^2 \), of the source representing the first-approximation spherical oscillations of the bubble \( B^j \).

Expressed in physical terms (velocities, pressures), the boundary condition (2) states that the first order correction, \( 0(\varepsilon) \), to the non-perturbed spherical behavior of the bubble \( B^i \) is a spherical modification of the collapse driving pressure. This would introduce, as for two bubbles, a spherical correction of the variations \( a^i_0(t) \). At the following orders new corrections of the uniform pressure appear, as well as a velocity field accounting for a slip velocity between the bubble and the surrounding fluid. Again, as in the two-bubble case, this induces a spherical correction and a nonspherical correction of the bubble shape. Therefore, one can show that the equation of the surface of the bubble \( B^i \) can be written in the form:

\[
R^i(\varepsilon^1g, t) = a^i_0(t) + \varepsilon a^i_1(t) + \varepsilon^2 \left[ a^i_2(t) + f^i_2(t) \cdot \cos \theta^i g \right] +
\]

\[
+ \varepsilon^3 \left[ a^i_3(t) + f^i_3(t) \cos \theta^i g + g^i_3(t) P_2(\cos \theta^i g) \right] + \ldots, \quad (3)
\]
Where the direction $\mathbf{B}^i\mathbf{C}^i$ (see figure 2) from which is measured the angle $\theta^i$ is compounded from all the $\theta^i_j$ and is obtained using Equation (7) presented below. Writing the nonspherical boundary conditions on the bubble wall and expanding $\theta_1$ in spherical harmonics one obtains the following differential equations for the successive corrections of $a^i_0(t)$ given by (1). Superscripts(i) are omitted for convenience:

\[
\begin{align*}
 a^1_0 \ddot{a}^1_0 + 3a^1_0 \dot{a}^1_0 + a^1 F (a^1_0, W_e, P_{g_0}, K) &= -\Sigma^1_j \left( \frac{\ell^1_0}{\ell^1_{ij}} \right) q^1_0, \\
 a^2_0 \ddot{a}^2_0 + 3a^2_0 \dot{a}^2_0 + a^2 F (a^2_0, a^2_1, W_e, P_{g_0}, K) &= -\Sigma^2_j \left( \frac{\ell^2_0}{\ell^2_{ij}} \right) q^2_1, \\
 a^3_0 \ddot{a}^3_0 + 3a^3_0 \dot{a}^3_0 + a^3 F (a^3_0, a^3_1, a^3_2, W_e, P_{g_0}, K) &= -\Sigma^3_j \left( \frac{\ell^3_0}{\ell^3_{ij}} \right) q^3_2, \\
 \vdots
 a^d_0 \ddot{a}^d_0 + 3a^d_0 \dot{a}^d_0 &= -\Sigma^d_j 3 \left( \frac{\ell^d_0}{\ell^d_{ij}} \right)^2 (\dot{a}^d_0 q^d_0 + a^d_0 \dot{q}^d_0), \\
 a^e_0 \ddot{a}^e_0 + 3a^e_0 \dot{a}^e_0 + 3F (a^e_0, a^e_1, a^e_2, W_e, P_{g_0}, K) &= -\Sigma^e_j 3 \left( \frac{\ell^e_0}{\ell^e_{ij}} \right)^2 (\dot{a}^e_0 q^e_0 + a^e_0 \dot{q}^e_0 + F q^e_2), \\
 a^f_0 \ddot{a}^f_0 + 3a^f_0 \dot{a}^f_0 - (\ddot{a}^f_0 - 6/W_e a^2_0) g^f_0 &= -\Sigma^f_j 5 \left( \frac{\ell^f_0}{\ell^f_{ij}} \right)^3 (a^2_0 q^f_0 + 2a^0_0 \dot{a}^f_0 q^f_0).
\end{align*}
\]

In these equations $F_0, F_1, F_2, F_3$ are known functions depending on the physical constants, $W_e$ and $P_{g_0}$, and on the calculated preceding orders of approximation. The deformations $f_2, f_3$ of the bubble $B^i$ and the motion of its center toward $B^j$; $\lambda_2, \lambda_3$; have been replaced by $d_2, d_3$ which indicate the total motion of the point $E_1$ toward $E_1$ (Figure 2).

\[
\ddot{d}_2 = \ddot{f}_2 - \dot{\lambda}_2 \; ; \; \ddot{d}_3 = \ddot{f}_3 - \dot{\lambda}_3
\]

When all the initial radii of the bubbles in the cloud are identical, the right-hand sides of Equation (4) are the same as those for the two-bubble case right-hand sides multiplied by one of the geometrical constants $c_1, c_2, c_3$:

\[
\begin{align*}
 c_1 &= \Sigma_j \left( \frac{\ell^i_0}{\ell^i_{ij}} \right), \\
 c_2 &= \Sigma_j \left( \frac{\ell^i_0}{\ell^i_{ij}} \right)^3 \cos \theta^i_{ij}, \\
 c_3 &= \Sigma_j \left( \frac{\ell^i_0}{\ell^i_{ij}} \right)^2 \sin \theta^i_{ij}.
\end{align*}
\]
where $a_o(t)$ is the nondimensional solution for a bubble of unit initial radius, the solution $a_o(t)$, for a bubble of normalized initial radius $\lambda$ is such that, $a_o(\lambda t) = \lambda a_o(t)$, the right-hand sides of (4) can be easily computed when $\lambda$s are known.

Indeed the whole problem can be reduced to the case of two interacting bubbles of different sizes. The comparison of equations (4) with those obtained in the case of two-bubbles shows that the $N$ bubbles in the cloud other than $B^1$ can be replaced by a unique bubble of strength $q_{n}^{i}g$, located at $G^1$, a distance $d_{o}^{i}g$ from $B^1$ in the direction defined by the angle $\theta_{n}^{i}g = \alpha_{i}^{g}$. As this equivalent bubble should induce the same pressures and velocities as defined by (2), its location and strength are obtained by the equations:

$$ q_{n}^{i}g/\theta_{o}^{i}g = \sum_{j=1}^{N} q_{n}^{j}/\theta_{o}^{j}, $$

$$ \alpha_{n}^{g} \cdot q_{n}^{i}g/(\theta_{o}^{i}g)^2 = \sum_{j=1}^{N} \alpha_{n}^{j} \cdot q_{n}^{j}/(\theta_{o}^{j})^2, $$

where $\alpha_{n}^{g}$ and $\alpha_{n}^{j}$ are respectively unit vectors of the directions $B^1G^j$ and $B^jG^1$ (Figure 1), and $n$ is the order of approximation. These equations define the angle $\theta_{n}^{i}g$, and the direction in which $d_{n}^{i}g(t)$ is measured in equation (4).

B. Pressure Field

For a given $P_{o}(t)$, equation (1) can be solved for the variations of the bubble radius, $a_{o}^{i}(t)$. This allows the subsequent determination of the pressure field around the bubble $B^1$, of center $B^1$, by the use of:

$$ P_{o}(B^1, r, t) = P_{o}(t) + (2 a_{o}^{i} a_{o}^{i} + a_{o}^{i} a_{o}^{i} \dot{a}_{o}^{i})/r - a_{o}^{i} a_{o}^{i} / 2r, $$

where $r$ is the distance between $B^1$ and a given point $M$ in the fluid.

The following corrections of $P_{o}$ are obtained once the successive orders of the problem are solved. The nondimensional outer problem, $\phi$, can be written:

$$ \phi(M, t) = -\sum_{i} \left[ q_{o}^{i} e^{i} + \frac{q_{o}^{i}}{r} e^{i} + \frac{q_{o}^{i}}{r^2} e^{i} + \epsilon^3 \left( \frac{q_{o}^{i}}{r^3} - \frac{h_{o}^{i}}{r^4} \cos^2 \theta_{o}^{i}g \right) + O(\epsilon^3) \right], \quad (8) $$

$$ c_{n} = \sum_{j} \left( \ell_{o}^{j} / \ell_{o}^{ij} \right)^3 \cdot P_{2} (\cos \theta_{ij}^{o}), \quad (6) $$

We can now compute the behavior of $B^1$ by solving the obtained differential equations (1 and 4) using a multi-Runge-Kutta procedure. The behavior of the whole cloud can then be obtained. This appears at first to be a very long task. However, noting that if $a_{o}(t)$ is the nondimensional solution for a bubble of unit initial radius, the solution $a_{o}(t)$, for a bubble of normalized initial radius $\lambda$ is such that, $a_{o}(\lambda t) = \lambda a_{o}(t)$, the right-hand sides of (4) can be easily computed when $\lambda$s are known.
where bars denote nondimensional "outer" quantities, and tildes nondimensional "inner" quantities.

\[ \ddot{\tilde{\phi}} = \tilde{\phi} \cdot T/r_{bo}^3 \cdot \tilde{q}_n, \quad \ddot{\tilde{q}}_n = q_n \cdot T/r_{bo}^3 \cdot \ddot{r} = r^2/l_o. \]  

(9)

T is the characteristic time of the bubble collapse and \( r^2 \) is the distance between a field point \( M \) and \( B^1 \). The Bernoulli equation enables one to calculate \( P \) using (8). We can write nondimensional:

\[ \tilde{p}(M,t) = \frac{p(M,t) - p_\infty(t)}{\Delta \rho} = -\varepsilon \frac{\partial \tilde{\phi}}{\partial x} - \frac{1}{2} \varepsilon^4 \frac{1}{\Delta \phi} \left| \Delta \tilde{\phi} \right|^2. \]  

(10)

\( \Delta \rho \) is the amplitude of the pressure driving the collapse and \( \tilde{t} = t/T \), where

\[ T = r_{bo} \sqrt{\rho/\Delta \rho}. \]  

(11)

In the following, we will consider as an illustration a uniform field of bubbles; any bubble has the same geometrical position relative to the others, and thus the same behavior. The general expression (8) simplifies considerably to become:

\[ \tilde{p}(M,t) = \left( \varepsilon \tilde{q}_o + \varepsilon^2 \tilde{q}_1 + \varepsilon^3 \tilde{q}_2 + \varepsilon^4 \tilde{q}_3 \right) \Sigma_1 \left( \frac{1}{r^2} \right) + \]  

\[ -\varepsilon^4 \tilde{q}_o \Sigma_1 \left( \frac{\cos^2 \theta}{r^2} \right) -\varepsilon^4 \tilde{q}_o \frac{1}{2} \left| \nabla \Sigma_1 \left( \frac{1}{r^2} \right) \right|^2 + O(\varepsilon^3). \]  

(12)

In this expression, the summations are geometrical constants similar to \( c_1, c_2, c_3 \) (6). Thus, once the dynamical bubbles behavior is known as well as their distribution the pressure field is determined.

c. Examples: Spherical Shell of Bubbles

As an illustration of the method presented above let us consider a distribution of bubbles centered on the surface of a sphere and which have the same position relative to each other. We will study the bubble behavior and the pressure generated for two types of ambient pressure variations with time: a) the classical case of a sudden positive pressure jump of amplitude \( \Delta \rho \), b) the case of a sudden pressure drop, \( \Delta \rho \), followed by a return to the initial pressure after a time period \( \Delta t \) during which the minimum pressure is kept constant.

In figure 3, the results of five different computations for case a), are compared, expansions being conducted up to \( \varepsilon^3 \). The ratio, \( \varepsilon = r_{bo}/l_o \), was kept constant and at a value of 0.05. The cases of two, three and twelve bubbles of centers located on the surface of a sphere are presented together with that of an isolated bubble. The
fifth case is an intermediate situation between the configurations of three and twelve bubbles. This case is arbitrary and is only determined by the choice of $c_1$, $c_2$, and $c_3$. In each case the variation with time of the distance, $B^1E^1$, (Figure 2) between the extreme point on a bubble $E^1$, and its initial center, $B^1$, is chosen to represent the bubble dynamics. Taking the bubble collapse in an unbounded fluid as reference, it is easy to see from Figure 3 how increasing the number of bubbles changes the dynamics of the one studied. We can observe first that, during the early slow phase of the implosion process, the collapse is significantly delayed. At any given nondimensional time the distance between $B^1$ and $E^1$ (and simultaneously the bubble characteristic size) is greater when the number, $N$, of interacting bubbles increases. Then, in the final phase of the implosion the tendency is reversed: the phenomenon speeds up and, in a shorter total implosion time, the final velocity of the motion is higher when $N$ increases. As we will see later, this effect can be easily explained by accounting for the modification of the driving pressure of the collapse of any bubble due to the dynamics of the other bubbles.

Figure 4 shows the behavior of the bubbles in the case of a pressure variation of type b. The cases of an isolated bubble and two, three, five and twelve bubbles are investigated again, and the variations of $B^1E^1$ with time are plotted. The ratio $\varepsilon$, and the duration $\Delta T$, of the pressure drop are kept constant and at the particular values of 0.1 and 0.8 respectively. Here, as in the preceding figure, noticeable changes can be observed when the degree of interaction increases. First, the growth is slowed down and retarded in comparison with the isolated case. Then, the collapse is accelerated and as a result the total implosion time decreases with an increase in the number of bubbles, $N$. While for $N = 2$, the total implosion time is greater than that of an isolated bubble, for $N = 12$ the time is significantly smaller. As we will see below this acceleration of the collapse makes the generated pressures at the end of the collapse higher than for the single bubble case.

Figure 5 compares for the same cloud configuration (twelve bubble, $\varepsilon = 0.1$) the bubble behavior for three values of the duration, $\Delta T$, of the pressure drop. The greater $\Delta T$ is, the longer the bubble is allowed to grow. As a result the maximum size it attains is bigger, but its lifetime is smaller. Thus, the resulting collapse is much stronger.

To examine the observations made above let us compare the imposed ambient pressure with the variations of the pressure generated at a distance $\lambda_0$ from a collapsing bubble in an infinite medium. As we can see from Figure 6, the perturbation pressure, i.e. the difference between the pressure at $\lambda_0$ and the far-field pressure, is negative for $t < 0.75$. As a result a fictitious bubble placed at the distance $\lambda_0$ from this spherical bubble will sense a less important and more gradual increase in the surrounding pressure. In the considered case, instead of a sudden nondimensional jump of the pressure from 0 to 1, $P$ surges only to 0.84, then rises slowly, not attaining 1 until $t > 0.75$. This would affect the bubble dynamics exactly as observed in Figure 3, namely a less violent start of the collapse. As a result, we find at
the end of this process a larger bubble than would be observed in an infinite medium. This, added to the fact that in the later stages \((t \approx 0.75)\) the driving pressure increases up to 2.25 times the far-field pressure, makes the subsequent end of collapse much more violent.

The same type of observation is made in the case of a finite-time pressure drop. In the first time period, \(T\), the pressure sensed at a distance \(L_0\) from the bubble center, \(B_0\), is higher than the imposed one. As a result a second fictitious bubble placed at this distance from \(B_0\) would have a slower growth during \(\Delta T\). This phenomena is however reversed in the second phase as an expansion wave is generated by the growing bubble \(B_0\). In the third and last phase a compression wave increases the driving pressure for collapse making this one more intense. In the presence of several bubbles the effects described above are amplified. Figure 7 is an example of this for the case of twelve bubbles. Plotted are the pressures generated during the bubble history at two locations: a) the center of the cloud and b) the center of one bubble, \(B^1\), in its absence. These pressures are compared with those generated during the growth and collapse of an isolated bubble at a distance equal to the spherical cloud radius. The corresponding bubble radius variation with time is that represented in Figure 5 (12 bubbles \(T = 0.6\)). The high pressure surge at the end of the collapse will be considered in the following.

Figure 8 is a collection of the results obtained in several cases studied. The maximum nondimensional pressure generated during the cloud collapse are represented versus the number of bubbles in the cloud. The cumulative effect is obvious since the values obtained vary in a several orders of magnitude range. The numbers represented should not be considered accurate since other scales for times, pressures and lengths are needed at the end of the collapse. Instead, they are presented here to give an indication of how tremendous pressures can be generated with an increasing number of interacting bubbles, and to give an idea of the trend of this increase. In this figure, the maximum pressures are given at the cloud center, \(C\), at the center of a bubble, \(B^1\), if it was removed, and at a distance \(r_{B_0}\) from \(B^1\).

The important role played by the gas content of the bubbles is to be emphasized. Increasing \(P_{B_0}\) from 0.1 to 0.2 reduces dramatically the generated pressures. This comes mainly from the fact that the cushioning effect of the gas reduces significantly the velocities attained at the end of the implosion.

Another very interesting observation from figure 8 is that the maximum pressures generated at the end of the collapse is much lower for a pressure drop of finite duration followed by a recompression in comparison with the pressure jump case. This effect is not due to the apparent higher gas content in this case. Indeed, the value of \(P_g\) to consider for comparison purposes should be for all cases the minimum gas pressure, \(P_{g\text{min}}\), which exists at the start of the collapse when the bubble has its maximum volume. For the case of twelve bubbles for example and a pressure drop (\(\Delta T = 0.8, P_{B_0} = 0.53\)) the value of \(P_{g\text{min}}\) is 0.07. The effective gas content is thus smaller, and the observed pressure drop is intrinsically related to the imposed pressure.
function.

This observed pressure attenuation can be explained by the fact that the cumulative effect of the other bubbles on the initial phase of the dynamics of the considered bubble is of opposite nature for the two pressure cases. In the pressure jump case the presence of other bubbles reduces initially the effective driving pressure of the bubble collapse thus preventing the bubble size from being small in the later phase when the collapse pressure surge occurs (see Figure 6). Conversely, the initial cumulative effect in the case of a finite time pressure drop is to reduce the bubble growth thus reducing the bubble size when the pressure surge occurs.

Another parameter on the value of the maximum pressure generated is the duration of the pressure drop. This effect is shown in Figure 9 for twelve bubbles and \( \varepsilon = 0.1 \). The previous type of reasoning when applied to the gas pressure leads us to believe that the increase of the maximum pressure with \( \Delta T \) is mainly due to a decrease in the effective initial gas content at the start of the collapse since the maximum bubble radius increases with \( \Delta T \).

III. CONTINUUM MEDIUM APPROACH

One major assumption of the theoretical approach as used in the preceding section is that, in first approximation the imposed ambient pressure is assumed to be instantaneously transmitted to the vicinity of each bubble in the cloud. Therefore, both the compressibility of the bubbly medium and the influence of the liquid motion generated by the other bubbles on the dynamics of the bubble considered were neglected in the first order approximation. This limits the validity of the study to very low void fractions. The incompressibility assumption is valid as long as the fluid velocity does not approach the speed of sound. For single bubble dynamics this does not usually happen until the final phase of the collapse. Here, however, two factors contribute to limit the validity of the assumption. First, the rate of implosion is higher and second, more important, the velocity of sound drops considerably when the void fraction increases. This underlines the need to account for the behavior of the cloud as a whole in order to determine a more accurate value of the local pressure driving the collapse of the individual bubbles. In addition this would have the advantage of limiting, for the following orders of approximations, the number of bubbles directly influencing the considered one. Indeed, the asymptotic theory shows that the effective parameter of the expansions is \( \varepsilon c_1 \) (where \( c_1 \), defined by (6), is a direct function of the number of bubbles), rather than \( \varepsilon = \frac{r_b^2}{\lambda_q} \). Introducing a motion equation for the bubbly medium would limit the number of influencing bubbles to those in the direct vicinity of the considered one, through a time delay of the propagation of the information from one bubble to another. In summary, if we account for a motion equation in the cloud medium the first order approximation of the preceding approach becomes more accurate and as a consequence the following corrections will be
smaller making the approach valid for higher void fractions, $\alpha$.

A. Classical Description

Basically the classical methods used to describe a two-phase medium are not much different from the singular perturbation method we presented above. The final description deals just with the macroscale of the cloud. However, this description is obtained by averaging the various physical quantities defined in the microscale. The two phase medium is assumed to be constituted of "particles" containing the host liquid and few bubbles. This "particle" is small enough to be able to distinguish the gaseous and liquid constituents, but large enough to enable one to define significant volume average quantities in the two-phase continuum. Therefore, each "particle" appears in the macroscale as a fluid point $M$ allotted various physical and kinematic properties: $\alpha(M,t)$ is the local void fraction, $\rho_m(M,t)$ is the local medium density, $U_m(M,t)$ is the velocity and $P_m(M,t)$ the pressure, .... etc. In such a volume averaging description, if $V_p$ is the volume of the particle, $X(M,t)$ the considered average quantity and $x(m,t)$ its local value in the microscale, we have the following definition:

$$X(M,t) = \frac{1}{V_p} \int_{V_p} x(m,t) \, dv$$  \hspace{1cm} (13)

The density of the medium is therefore defined by the relation:

$$\rho_m(M,t) = \rho_l [1 - \alpha(M,t)] + \rho_g(M,t) \alpha(M,t) , \hspace{1cm} (14)$$

where $\rho_g(M,t)$ is defined by (13). The liquid is assumed to be incompressible and $\rho_l$ constant. The void fraction, $\alpha(M,t)$, is defined as the relative volume of gas in the particle. Usually $\rho_g \alpha$ is neglected and the density of the medium is written:

$$\rho_m(M,t) \approx \rho_l [1 - \alpha(M,t)] . \hspace{1cm} (15)$$

If $U_l(M,t)$ is the average velocity of the liquid in the particle and $U_g(M,t)$ the average velocity of the gas, we obtain comparable results to (14) and (15):

$$\rho_m U_m = \rho_l U_l (1 - \alpha) + \rho_g U_g \alpha , \hspace{1cm} (16)$$

$$\rho_m U_m \approx \rho_l U_l (1 - \alpha) \hspace{1cm} , \hspace{1cm} (17)$$

and combining with (15),

$$\frac{U_m}{U_l} \approx 1 . \hspace{1cm} (18)$$
The continuity equation is obtained by writing the mass conservation of a volume of the bubbly medium followed during its motion. Using the average quantities defined above we can write:

\[
\frac{d}{dt} \int_{V(t)} \rho_m \, dV = \int_{V(t)} \left[ \frac{d\rho_m}{dt} + \rho_m \nabla \cdot \overrightarrow{U_m} \right] \, dV = 0 .
\] (19)

Here the material derivative pertains to the medium velocity \( \overrightarrow{U_m} \), or with our assumptions to \( \overrightarrow{U_l} \) (see (18)).

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \overrightarrow{U_l} \cdot \nabla .
\] (20)

As Equation (19) is valid for any volume \( V \), we obtain the general equation:

\[
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \overrightarrow{U_m}) = 0 ,
\] (21)

where \( \rho_m \) is defined by either (14) or (15).

A similar equation can be written concerning the number of bubbles, \( n(M,t) \). Neglecting any complete bubble disappearance or sudden generation, as well as bubble splitting and coalescence we can write:

\[
\frac{Dn}{Dt} + n \nabla \cdot \overrightarrow{U_g} = 0 ,
\] (22)

the material derivative being defined as:

\[
D/\partial t = \partial/\partial t + \overrightarrow{U_g} \cdot \nabla .
\] (23)

The momentum equation of the bubbly medium can be obtained in the same manner by using the momentum equations of both constituents in the microscale and integrating over the "particle" volume \( V_p \). If we neglect the viscous forces, this can be written:

\[
\int_{V_p} \left[ \rho_i \frac{dU_i}{dt} + \nabla P_i \right] \, dV = 0 ,
\] (24)

the index \( i \) designating the liquid or the gaseous phase depending on the position of the element of volume \( dV \) in the microscale. If we account for the incompressibility of the liquid this equation becomes:

\[
\rho_l (1 - \alpha) \frac{dU_l}{dt} + \rho_g \frac{DU_g}{Dt} - \rho_g \int_{V_p} \overrightarrow{U_g} \cdot \nabla \overrightarrow{U_g} \, dV + \int_{V_p} \nabla P_i \, dV = 0 .
\] (25)

If we neglect the gas contribution to the momentum, and we account for
(18) we obtain the following approximate classical momentum equation:

$$
\rho_e (1 - \alpha) \frac{dU}{dt} + \nabla \cdot p_m = 0 ,
$$

(26)

where it is assumed that

$$
\int \nabla p_1 \cdot dU = \int_A p_1 n \cdot ds \approx \int_p \nabla \cdot p_m .
$$

(27)

The only equation left is that giving the bubble translation velocity, $U_g$, which reflects the interaction between the two phases of the bubbly medium. The study of this equation is a whole subject of research in itself. Several contributions exit which have dealt with more and more complicated situations. To quote some without trying to be extensive we can add to the above references Johnson and Hsieh (1966), Landweber and Miloh (1980), Van Wijngaarden (1976, b), Van Beek (1981). When viscous drag is neglected a very interesting general expression for the motion of a deformable bubble in a nonuniform potential flow was derived by Landweber and Miloh (1980). If we admit, however, that the liquid flow around an isolated bubble is linearly accelerated, and that the bubble remains in first approximation spherical, we can write, neglecting the bubble mass, a simpler equation as follows:

$$
\frac{DU_g}{Dt} - 3 \frac{DU_g}{Dt} = 3 \frac{\dot{a}}{a} (U_g - U_g) .
$$

(28)

In this equation the virtual mass of the bubble is considered to be $2/3 \pi a^3 \rho_g$ and the material derivative is related to the bubble velocity as discussed by Prosperetti and Van Wijngaarden (1976).

When other bubbles are present in the flow corrections are to be introduced in this expression, following Landweber's calculations. Van Wijngaarden (1976, a) and Van Beek (1981) performed similar corrections for a rigid sphere and obtained the expression:

$$
\frac{d}{dt} \left[ \frac{1}{2} (1 + \zeta) (U_g - U_g) \right] = (1 - \alpha) \frac{d}{dt} U_g .
$$

(29)

where $\zeta$ is a correction to the added mass of the sphere due to the presence of the cloud. They gave, however, respectively the values (2.78 $\alpha$) and (-0.225 $\alpha$) for $\zeta$.

B. Micromorphic Continuum Description

In classical continuum mechanics the fluid is described geometrically by a field point $M$ and kinematically by a velocity field $U(M)$. The averaging approach of the cloud medium, as described in the preceding paragraph, is in this sense classical. However, when a medium contains microstructure, as is the case for a bubbly medium, a more refined description can be obtained by assigning to $M$, in
addition to the macroscale velocity, \( U(M) \), other quantities which reflect the microscale behavior in the "particle". In a first gradient theory, in addition to the velocity field, \( U(m) \), a field of the gradients of relative velocities in the microscale scale, \( X \), is added which defines kinematically the medium*. The description can be further refined by using higher order gradient theories. Germain (1973) considered such approaches and, using the method of virtual power, was able to derive the equation of motion of the continuum medium accounting for the macrostresses, \( \sigma \), and the microstresses, \( \bar{S} \).

In a first gradient theory the velocity in the microscale can be written as

\[
U'(m) = U(M) + X(M) \cdot Mm.
\]  

(30)

Consequently the acceleration, \( \Gamma' \), of \( m \) is derived and, by equating at dynamical equilibrium the virtual power of all the internal and external forces acting on the considered particle (volume \( V_p \)) to the material derivative of the virtual power of mass velocity of \( V_p \), one obtains a dynamical equation of the medium relating \( \bar{S} \), \( \vec{a} \), and \( \Gamma' \).

To define \( X \) we consider the motion on a scale which is of the same order as the microstructure. To do so for a bubble cloud, let us divide the cloud medium into fluid "cells" each enclosing an isolated bubble. In addition, we assume for simplicity that the bubble center of mass and the "cell" center of mass coincide at the considered time. Let \( U(M) \) be the velocity in \( M \) induced by the rest of the cloud in absence of the bubble, and \( V(B) \) the velocity of the bubble center, \( B \). \( U(M) \) would be the value of the velocity field assigned to \( M \) in a classical fluid mechanics description.

The bubble radius is \( a_0 \) and its variations with time are given by (1). This radial motion of the bubble surface induces at a point \( m \) of the cell (Figure 10) a velocity of value \( (u_r \cdot \epsilon_r') \), where \( \epsilon_r' \) is the unit vector of the direction \( Mm \). The total velocity \( u' \), at \( m \) is:

\[
u'(m) = U(M) + \frac{\dot{a}_0}{r'^2} \cdot \epsilon_r' + \nabla \left[ \frac{a_0^3}{2r'^2} (U(M) - V(B)) \cdot \epsilon_r' \right] + \ldots (31)
\]

where \( r' \) is the distance between \( M \) and \( m \). The second term in this expression is a source term due to the spherical bubble oscillations, while the last term, \( u'' \), is a dipole due to the slip velocity between the spherical bubble and the fluid, and could include first order corrections of the bubble shape. For further corrections for nonsphericity of the bubble, other terms (singularities of higher orders) have to be included. By differentiating (31) with respect to time and space one can define an acceleration vector, \( \Gamma'' \), and a strain rate tensor, \( D'' \). Following Germain's approach, and using the principle of virtual powers, one could then derive an equation of motion of the cloud medium. We decided instead to start with a first gradient

* A double underlined quantity is a tensor.
theory and replace (31) by its Taylor expansion. We follow in doing so
the first calculations done by Michelet (1980) in his graduate thesis.

The basic approximation used in this linearization approach is
based on the fact that Equation (31) is only valid in the liquid
portion of the "cell" (r ≫ a°). It seems therefore logical to write
the velocity in m, close to the bubble boundary, as a Taylor expansion
of the value of u' computed on a point of the bubble surface, S,
(Figure 10). This has the advantage of eliminating the singularity of
(31) for r' = 0. The obtained expression for u'(m) is then:

\[
\begin{aligned}
u'(m) &\approx U(M) + \left[ 3a_0 + 4V_t \cos \theta - r' \left( 2 \frac{\dot{a}_0}{a_0} + 3 \frac{V_t}{a_0} \cos \theta \right) \right] \frac{\epsilon_r}{r} + \\
&+ \left[ 2V_t \sin \theta - \frac{3}{2a_0} r' V_t \sin \theta \right] \frac{\epsilon_\theta}{\eta},
\end{aligned}
\]  

(32)

where V_t = | V(B) - \overline{U}(M) |.
When V_t is not accounted for, the expression of u'(m) reduces to a
form comparable to (30), which is much easier to interpret than equa-
tion (32). In that case we obtain:

\[
\begin{aligned}
u'_0(M') &\approx U(M) + \overline{x} \cdot \overline{Mm} + \overline{a} \cdot \frac{\epsilon_r}{r'},
\end{aligned}
\]  

(33)

where \( \overline{x} \) and \( \overline{a} \) are both tensors assigned to M and defined as:

\[
\begin{aligned}
\overline{x} &= -2 \frac{\dot{a}_0}{a_0} I, \quad \overline{a} = 3 \frac{a_0}{a_0} I.
\end{aligned}
\]  

(34)

We notice that in comparison to (30), which describes a first grad-
ient homogeneously deformation, in (33) there is in addition to the
gradient tensor, \( \overline{x} \), a tensor \( \overline{a} \) reflecting the presence of a source in
the cell. Equation (32) reflects in addition to this the presence of
a dipole. It could be written as

\[
\begin{aligned}
u'(M') &\approx U(M) + \overline{x} \cdot \overline{Mm} + \overline{x}' \cdot \frac{\overline{Mm}}{r} \frac{\epsilon_z}{z} + \frac{\epsilon_r}{r} + \frac{\epsilon_r'}{r'} \overline{\epsilon_z} \\
&+ \left( 6V_t \cos \theta - \frac{9}{2a_0} r' V_t \cos \theta \right) \frac{\epsilon_r}{r'},
\end{aligned}
\]  

(35)

where \( \overline{\epsilon_z} \) is the unit vector of the direction of U and V; \( \overline{x}' \) and \( \overline{a}' \) play
the same role as \( \overline{x} \) and \( \overline{a} \) but are applied just to the direction of the
translation. The last inhomogeneous term is more difficult to put
in simple form.

From the expression (32) we can now compute the acceleration,
then apply the principle of virtual power, to obtain the equation of
motion. Here again in absence of translation velocity $V_t$, the results are simpler to interpret. In absence of viscous effects these results can be written as follows:

$$\rho_m \left[ \frac{dU}{dt} + 3 K \left( \frac{a_o}{a_0} - 2 \frac{\dot{a}_o^2}{a_0} \right) \right] = -\nabla p,$$  \hspace{1cm} (36)

where $K$ depends unfortunately on the cell geometry,

$$\rho_m K = \int_V \rho_i e_r \, dV.$$  \hspace{1cm} (37)

If the cell and the bubble are symmetrical with regard to the center of mass $M$, then $K = 0$, and (37) reduces to the classical equation of motion, (26). Although it is unfortunate that the cell shape seems to play a role in the model, $K$ might rather reflect an effect of the non-sphericity of the bubble.

When $V_t$ is taken into account a whole series of "inertia" integrals like (37) appear in the calculations. In order to see what such a model might indicate we considered the case of a spherical bubble in a spherical cell. In this case the motion equation becomes:

$$\rho_m \left\{ \frac{dU}{dt} + \frac{\dot{a}_o}{a_0} \left[ 3 + \frac{3}{4} \frac{R}{a_0} - 6 \frac{a_o}{R} + O(\alpha) \right] (V - U) \right\} = -\nabla p.$$  \hspace{1cm} (38)

Here, $R$ is the radius of the cell, and if we write $R = a_o \alpha^{-1/3}$, we have the unusual result:

$$\rho_m \left\{ \frac{dU}{dt} + 3 \frac{\dot{a}_o}{a_0} \left[ \alpha^{-1/3} \left( \frac{1}{4} + \alpha^{1/3} - 2 \alpha^{2/3} + \ldots \right) \right] (V - U) \right\} = -\nabla p$$  \hspace{1cm} (39)

This surprising result (dependence on $\alpha^{-1/3}$) might be compared with that obtained for the apparent viscosity of a bubbly flow, which is $4\mu/3 \cdot \alpha^{-1}$. (Batchelor (1967), Van Wijngaarden (1972)). We recognize however that the present model is in its infancy and should be carefully checked before any conclusions are drawn. In addition, due to the linearization of the velocity field (first gradient theory) this model loses its validity for low $\alpha$.

C. Case of a spherically symmetrical cloud

Let us consider a finite size spherical cloud of bubbles and define its radius, $R(t)$, at time $t$, as the position of the last outer shell of bubbles. The space is therefore divided into two regions.
For $r > R(t)$, the medium is an incompressible liquid of density $\rho_\ell$, while the interior of the sphere, $r \leq R(t)$, is filled with a two-phase medium which can be defined as in the preceding paragraphs. Let us consider here the classical approach and define at a point $M(r)$, a radial liquid velocity $u_\ell (r,t)$ and a radial bubble translation velocity $u_\ell (r,t)$. Similarly we define a local void fraction $\alpha (r,t)$, density $\rho_m (r,t)$, bubble radius $a_0 (r,t)$, number density $n (r,t)$, and medium velocity $u_m (r,t)$. The matching between the two media, states that at $r = R(t)$ there is continuity of velocities and pressures:

$$\dot{R}(t) = u_\ell (r,t) \quad ,$$

$$P_\ell (R,t) = \lim_{r' \to \infty} p' (R, r', t) \quad ,$$

where $r'$ is the distance in the microscale between a bubble center and a cell field point. The continuity and momentum equations in the liquid medium ($r > R(t)$) are easy to solve and give, after neglecting viscous effects:

$$u_\ell (r) = \frac{\dot{V}}{4\pi r^2} \quad ,$$

$$\frac{\partial P_\ell}{\partial r} = - \frac{\rho_\ell}{4\pi} \left[ \frac{\ddot{V}}{r^2} - 2 \frac{\dot{V}^2}{r^3} \right] .$$

$V_g$ is the total volume of the bubbles in the cloud

$$V_g = 4\pi \int_0^R \alpha \ r^2 \ dr \quad (43)$$

Inside the bubbly medium, due to the spherical symmetry, the continuity equation also gives

$$u_\ell (r,t) = \frac{\dot{V}(r)}{4\pi r^2} \quad ; \quad r < R(t) \quad ,$$

with

$$V(r) = 4\pi \int_0^x \alpha (x,t) \ x^2 \ dx \quad .$$

If we are interested in the problem of the collapse of the cloud under an imposed ambient pressure variation, $P_\infty (t)$, (41) can be integrated between the cloud radius and infinity to give:

$$-P_\infty (t) + P_\ell (R) = \frac{\rho_\ell}{4\pi} \left[ \frac{\dot{V}_g}{R} - \frac{1}{2} \frac{\dot{V}^2}{g^4} \right] .$$

(46)
Using (40), \( P_\infty(R) \) can be related to the behavior of any individual bubble of radius \( a_0 \) in the last outer shell of the cloud, using equation (1). Equation (43) becomes:

\[
-P_\infty(t) + P_v + P_g \left( \frac{a_0}{a} \right) \frac{3k}{4\pi} \frac{\dot{a}}{R} - \frac{1}{2} \frac{\dot{R}^2}{R^2} + a_0 \ddot{a} + \frac{3}{2} \dot{a}_0^2 \right].
\] (47)

The cloud radius motion can be obtained by using an equation of the bubble motion, for instance (25) or (26). In the simplest case, equation (25) gives the following second relation between \( \dot{R}, a_0 \) and \( V_g \):

\[
\ddot{R} + 3 \frac{\dot{a}_0}{a_0} \dot{R} = \frac{3}{4\pi R^2} \left[ \dot{V}_g - \frac{\dot{V}_g^2}{2R^3} + \frac{\dot{a}_0}{a_0} \dot{V}_g \right].
\] (48)

A third equation, in addition to (47) and (48), is needed to solve for \( \dot{R}, a_0 \) and \( V_g \). Without an assumption on a proportionality between \( V_g(t) \) and \( a_0(t) \) without penetrating the cloud and solving for all \( a_0(r,t) \) to determine \( V_g \) there is no hope of solving the problem. We do not think the proportionality assumption is generally justifiable even if at \( t = 0 \) all bubbles in the cloud have the same size, since \( P(r,t) \) would not generally be the same for any location \( r \) at a subsequent time. This need to solve the whole problem is to be expected and is very important because it shows that defining the cloud by just one parameter, as a unique void fraction, is not sufficient to describe its dynamics. Number and bubble size distribution are other important variables to consider. An exception to this reasoning is the case of a cloud which possesses a high enough void fraction in order for a shock wave to form at \( \dot{R}(t) \) and separate the two media described here. Such an interesting model has been described by Mørch (1982).

IV. CONCLUSIONS

We have considered in this paper the collapse of a cloud of bubbles submitted to a change in the ambient pressure. Two types of models were presented. The first model, valid for low void fraction is an asymptotic approach based on the fact that the bubble radius is small compared to its distance from neighboring bubbles. This single perturbation method allowed us to write a system of differential equations which enables one to describe any bubble motion and deformation knowing the geometrical and size distributions of the bubbles. As a consequence the whole flow and pressure field can be determined. As an illustration a few cases of symmetrical bubble distributions on a spherical shell were considered and showed interesting results. Even for very low void fractions, collective bubble collapse can generate pressures orders of magnitude higher than those produced by single bubble collapse. This would tend to explain the observed high erosion intensities and the bending of trailing edges. The cumulative effect
comes from the fact that the interaction increases the driving pressure of collapse of each individual bubble. This augments the violence of its implosion and thus the interaction with the other bubbles. Thus, each bubble ends its collapse not under the effect of a pressure of the same order as the ambient, but orders of magnitude higher. This cumulative effect would not exist if the void fraction is high enough for the cloud to behave as a single bubble. The study showed again the importance of gas content in the bubble and the history of the ambient pressure variations.

The second approach is a continuum approach and is undertaken in order to extend the validity of the study to higher void fractions. We principally pointed out the difficulties and suggested a way of improving the averaging methods by accounting for the singular nature of the bubbly medium under collapse conditions. To do this we used a first gradient theory for the flow field and a micromorphic structure for the bubbly medium. A correction of order $\alpha^{-1/3}$ appears in the motion equation of the bubbly medium when the bubble radial oscillation and translation velocity are not negligible. We showed finally for a spherical cloud, with a classical continuum medium approach that it is not possible to easily solve the problem without imposing an assumption of a relationship between the behavior of the total gas volume in the cloud and that of an individual bubble. The knowledge of the local behavior of the bubbles in the cloud and thus of the local characteristics of the cloud (i.e., void fraction, bubble number density) seems necessary for solving the problem.

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V. REFERENCES


