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A NUMERICAL MODEL FOR THREE-DIMENSIONAL BUBBLE DYNAMICS
IN COMPLEX FLOW CONFIGURATIONS

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ABSTRACT

In most practical configurations where cavitation occurs, bubbles are not in a uniform or axisymmetric flow field and existing bubble dynamics models, either spherical or axisymmetric, are only more or less appropriate approximations. In this paper we will describe ongoing studies which consider the fully three-dimensional bubble dynamics problem. The interaction between a growing, detorming and collapsing bubble near a boundary and/or in a non-uniform flow field is simulated numerically using a Boundary Integral Method. The collapse of a large bubble near a solid flat plate in a gravity field is considered as a first example. The plate orientation relative to the gravity field significantly influences the three-dimensional bubble shape and behavior. Another not previously solved case considered here is the growth and collapse of a bubble in a vortex flow. The paper presents the method and shows on examples the influence of the various geometric or flow parameters on the bubble dynamics.

INTRODUCTION

Due to the complexity of the problem, most classical cavitation bubble dynamic studies have neglected bubble interaction and deformation and based their approach on isolated spherical bubble dynamics. However, with the recent advent of computational techniques and facilities, considerable attention has been given to the study of nonspherical bubble dynamics, particularly in the vicinity of a solid wall or a free surface [1-6]. All these studies were restricted to the case where external forces act in the same direction as the nearly rigid or free boundary and took advantage of the axisymmetry of the resulting problem. Deviations from this simplifying assumption were not considered, even though they are expected to significantly influence the results, for instance in the case of bubble dynamics in a boundary layer near a solid wall. To address this general problem we are developing a fully three-dimensional approach [7] which enables the investigation of bubble dynamics in complex flow fields such as vortical, boundary and shear flows.

One of the numerical methods that has proven to be very efficient in solving this type of free boundary problem is the Boundary Integral Method. References [2], [5], and [9] used this method in the solution of axisymmetric problems of bubble growth and/or collapse near boundaries. In this paper, a three-dimensional numerical code using the same method is presented. The growth and collapse of a large buoyant bubble near boundaries is shown as an example.

The three-dimensionality of the problem arises from orienting the rigid boundary arbitrarily relative to the direction of gravity. We present here an outline of the model, and some examples of bubble behavior in the presence of various solid boundaries. The detailed mathematical expressions involved are very cumbersome and are kept to a minimum. The interested reader is referred to Perdue [8] for more detailed descriptions and expansions.

In a second part of the paper the model is extended to the case where the basic flow (flow in absence of the bubble) is not restricted to being inviscid and potential and has only to satisfy the Navier Stokes equations. It is then shown that, as long as the flow due to the bubble dynamics can be assumed to be potential, the same method can be applied, provided a modified Bernoulli equation is used for the computation of the pressure field. The method is illustrated for the relatively simple case of bubble growth and collapse at the center of a linear vortical structure.

BUBBLE DYNAMICS EQUATIONS

It is widely accepted that due to the relatively large velocities involved, viscosity has no appreciable effects on the growth and collapse of cavitation bubbles in water. Also, since throughout most of the life of the bubble, the motion of the bubble wall is relatively slow compared to the speed of sound in water, compressibility effects can be ignored. This is valid until the latest collapse phase. These classical assumptions result in a flow due to bubble dynamics that is potential (velocity potential, \( \phi \)) and satisfies the Laplace equation,

\[
\nabla^2 \phi = 0.
\]

(1)

The solution must in addition satisfy boundary conditions at infinity, at the bubble wall and at the boundaries of any nearby bodies.

At all moving or fixed surfaces (such as the bubble surface or a nearby boundary) an identity between fluid velocities normal to the boundary and the normal velocity of the boundary itself is to be satisfied. For instance, at the bubble-liquid interface, the normal velocity of the moving bubble wall must equal the normal velocity of the fluid, or

\[
\nabla \phi \cdot \mathbf{n} = V_s \cdot \mathbf{n},
\]

(2)

where \( \mathbf{n} \) is the local unit vector normal to the bubble surface and \( V_s \) is the local velocity vector of the moving surface. For a plane solid wall, this condition is met exactly by including an
bubble size variation period. New coordinate positions of the nodes can then be obtained using the position at the previous time step and the displacement \( \delta \mathbf{V}_n = \mathbf{V}_n - \mathbf{V}_n \) where \( \mathbf{V}_n \) is the unit tangent vector. This time stepping procedure is repeated throughout the bubble growth and collapse, resulting in a shape history of the bubble.

PRESENCE OF A BASIC FLOW

Cavitation bubbles seldom grow and collapse in a quiescent fluid or in a uniform flow field. To the contrary, cavities are most commonly observed in shear layers, boundary layers and vortical structures. For the analysis of cavitation erosion and noise in flowing systems, the combination of a nonuniform flow field and the presence of a solid boundary is essential. To this end, let us consider the case where the basic flow is known and satisfies the Navier-Stokes equations:

\[
\frac{\delta \mathbf{V}_b}{\delta t} + \mathbf{V}_b \cdot \nabla \mathbf{V}_b = -\frac{1}{f} \nabla P_b + \nu \nabla^2 \mathbf{V}_b \quad .
\] (10)

In presence of the oscillating bubbles, the velocity field is given by \( \mathbf{V} \) which also satisfies the Navier-Stokes equation:

\[
\frac{\delta \mathbf{V}}{\delta t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{1}{f} \nabla P + \nu \nabla^2 \mathbf{V} \quad .
\] (11)

Bubble flow velocity and pressure variables, \( \mathbf{V}_b \) and \( P_b \), can be defined as follows:

\[
\mathbf{V}_b = \mathbf{V} - \mathbf{V}_o \quad , \quad P_b = P - P_o \quad .
\] (12)

If we assume that the bubble flow field is potential

\[
\mathbf{V}_b = \Phi_b \quad , \quad \nabla^2 \Phi_b = 0 \quad ,
\] (13)

and subtract (10) from (11) we obtain

\[
\frac{\delta \Phi_b}{\delta t} + \frac{1}{2} \left| \mathbf{V}_b \right|^2 + \mathbf{V}_b \cdot \nabla \mathbf{V}_b + P_b = \mathbf{V}_o \times \left( \nabla \times \mathbf{V}_o \right) \quad .
\] (14)

This equation, once integrated, is to be compared with the classical unsteady Bernoulli equation. As an illustration consider a wall boundary layer flow such that

\[
\mathbf{V}_o = f(z) \hat{\mathbf{c}}_x \quad ,
\] (15)

where \( z \) is the direction perpendicular to the wall and \( \hat{\mathbf{c}}_x \) is the unit vector in the flow direction. Then

\[
\nabla \times \mathbf{V}_o = \frac{\partial f}{\partial z} \hat{\mathbf{c}}_y \quad .
\] (16)

Substituting (16) in (14) and computing the scalar product with the unit vector in the \( y \) direction, \( \hat{\mathbf{c}}_y \), one obtains along a path parallel to the \( y \) axis:

\[
\psi = \frac{\delta \Phi_b}{\delta t} + \frac{1}{2} \left| \mathbf{V}_b \right|^2 + \mathbf{V}_b \cdot \mathbf{V}_b + \frac{P_b}{f} = \text{constant}
\] (17)

A similar result is obtained when the basic flow field is that of a two-dimensional Rankine vortex, \( \mathbf{V}_o = \mathbf{V}_o e_y \):

\[
\begin{align*}
\mathbf{V}_b &= \frac{\Gamma}{(2\pi r)} \quad ; \quad r \geq a_c \\
\mathbf{V}_b &= \omega r = \frac{\Gamma}{(2\pi a_c^2)} \quad ; \quad r \leq a_c,
\end{align*}
\] (18)

where \( \Gamma \) is the vortex circulation and \( \mathbf{V}_b \) the tangential velocity. If the bubble is located on the vortex axis, then we can assume that \( \mathbf{V}_b \) has only axial and radial components and then obtain in cylindrical coordinates:

\[
\begin{align*}
\frac{\delta \psi}{\delta r} &= 0 \\
\frac{1}{r} \frac{\delta \psi}{\delta \theta} &= 2 \omega r \\
\frac{\delta \psi}{\delta z} &= 0.
\end{align*}
\] (19)

In that case the Bernoulli equation is to be replaced by:

\[
\frac{\delta \Phi_b}{\delta r} + \frac{1}{2} \left| \mathbf{V}_b \right|^2 + \frac{P_b}{f} = \text{constant along radial directions}
\] (20)

Accounting for at-infinity conditions, the pressure at the bubble wall \( P_w \) (see equation (15)) is related to the pressure field in the Rankine vortex \( P_o \) and to the bubble flow field by:

\[
\begin{align*}
P_w &= \frac{\delta \Phi_b}{\delta t} - \frac{1}{2} \left| \mathbf{V}_b \right|^2 \\
\left[ \frac{f}{\delta t} - \frac{1}{2} \right] P_o &= \text{constant at bubble wall}
\end{align*}
\] (21)

The nondimensional basic flow pressure is known and given by:

\[
\begin{align*}
P_w(\tilde{r}) &= \frac{1}{\Omega} \left[ 1 - \frac{1}{2} \left( \frac{\tilde{r}}{\tilde{s}_c} \right)^2 \right] \quad ; \quad \tilde{r} \leq \tilde{s}_c, \\
P_w(\tilde{r}) &= \frac{1}{\Omega} \left( \frac{\tilde{s}_c}{\tilde{r}} \right)^2 \quad ; \quad \tilde{r} \geq \tilde{s}_c
\end{align*}
\] (22)

where pressures are normalized with the ambient pressure, \( P_w \) and lengths by the maximum radius, \( R_{max} \), the bubble would achieve if the pressure drops in an infinite medium to the value on the vortex axis. The swirl parameter \( \Omega \), defined as

\[
\Omega = \frac{f}{P_o} \left( \frac{\Gamma}{2\pi a_c} \right)^2
\] (23)

characterizes the intensity of the pressure drop due to the rotation relative to the ambient pressure. The pressure on the vortex axis is \( (1-\Omega) \) and goes to zero if \( \Omega=1 \).

COMPUTATIONAL RESULTS AND DISCUSSION

The numerical code was tested against simple known results in the literature such as the collapse of a spherical bubble. With a discretized bubble of 162 nodes, the observed errors were less than 0.14 percent and dropped to 0.05 percent for 252 nodes. The 162 discretization nodes (320 triangular panels) was selected for most of the runs shown below. Comparisons were also made with the previously studied axisymmetric cases available in the literature, and have shown.
image bubble in the computation of the potential and its normal derivative over the bubble surface.

The bubble is assumed to contain noncondensible gas as well as vapor of the surrounding liquid. The pressure within the bubble at any given time is considered to be the sum of the partial pressures of the noncondensible gases,  \( P_g \), and the vapor,  \( P_v \), inside the bubble. Vaporization of the liquid is assumed to be fast enough so that the vapor pressure remains constant throughout the simulation and equal to the equilibrium vapor pressure at the ambient liquid temperature. To the contrary, gas diffusion does not have time to occur and the noncondensible gases are assumed to have a polytropic behavior,  \( P_v^k = \text{constant} \), where  \( V \) is the bubble volume and  \( k \) is the polytropic constant varying from  \( k = 1 \) for isothermal behavior to  \( k = 1.4 \) for adiabatic conditions.

The pressure at the bubble surface,  \( P_L \), is obtained at any time from the following pressure balance equation:

\[
P_L = P_v + P_g (\delta) - C \sigma \tag{3}
\]

where  \( P_{vg} \) and  \( V_g \) are the initial gas pressure and volume respectively,  \( \sigma \) is the surface tension,  \( C \) is the local curvature of the bubble, and  \( V \) is the instantaneous value of the bubble volume.  \( P_{vg} \) and  \( V_g \) are known quantities at  \( t=0 \).

**NUMERICAL METHOD IN ABSENCE OF BASIC FLOW**

In order to simulate bubble behavior in complex geometry and flow configurations, a three-dimensional Boundary Integral Method is chosen. This method is based on Green's equation which effectively reduces by one the dimension of the problem. If the velocity potential,  \( \Phi \), and its normal derivatives are known on the fluid boundaries (points M) and  \( \Phi \) satisfies the Laplace equation, then  \( \Phi \) can be determined anywhere in the domain of the fluid (field points P). This can be written:

\[
\int_S \left[ \frac{\delta \Phi}{\delta n} \left( \frac{1}{|MP|} \right) + \frac{\delta}{\delta n} \left( \frac{1}{|MP|} \right) \right] dS = \pi \Phi (P) \tag{4}
\]

where  \( \pi \Phi = \Phi \), the solid angle under which P sees the fluid.

a = 4, if P is a point in the fluid,

a = 2, if P is on a smooth surface, and

a < 4, if P is a point at a sharp corner of the surface.

If the field point P is selected to be on the surface of the bubble or its image, then a closed set of equations can be obtained and used at each timestep to solve for values of  \( \delta \Phi / \delta n \) (or  \( \Phi \)) assuming that all values of  \( \Phi \) (or  \( \delta \Phi / \delta n \)) are known at the preceding step.

To solve Equation (4) numerically, it is necessary to discretize the bubble into panels, perform the integration over each panel, and then sum up the contributions to complete the integration over the entire bubble surface. To do this, the initially spherical bubble is discretized into a geodesic shape with flat, triangular panels. With the discretized surface, Equation (4) becomes a set of N equations (N is the number of discretization nodes) of index i of the type:

\[
\sum_j \left( \frac{\delta \Phi_j}{\delta n} A_{ij} \right) = \sum_j \left( B_{ij} \Phi_j \right) - \pi \Phi_i \tag{5}
\]

where  \( A_{ij} \) and  \( B_{ij} \) are elements of matrices which correspond numerically to the integrals given in Equation (4).

To evaluate the integrals in (4) over any particular panel, a linear variation of the potential and its normal derivative over the panel is assumed. In this manner, both  \( \Phi \) and  \( \delta \Phi / \delta n \) are continuous over the bubble surface and are expressed as a function of the values at the nodal points which define the particular panel. Obviously higher order expansions are conceivable, and would improve accuracy at the expense of additional analytical and numerical computation times. The two integrals in (4) are then evaluated analytically and the resulting expressions, too long to present here, can be found in [8].

In order to proceed with the computation of the bubble dynamics several quantities appearing in the above boundary conditions need to be evaluated at each time step. The bubble volume presents no particular difficulties, while the unit normal vector, the local surface curvature and the local tangential velocity at the bubble interface need further development. The curvature of the bubble surface is obtained by first computing a local bubble surface three-dimensional fit, \( \{x,y,z\} = 0 \). The unit normal at a node can then be expressed as:

\[
\hat{n} = \pm \bar{\nabla} f / |\bar{\nabla} f| \tag{6}
\]

The appropriate sign is chosen to insure that the normal is always directed toward the fluid. The local curvature is then computed by:

\[
C = \bar{\nabla} \times \hat{n} \tag{7}
\]

To obtain the total fluid velocity at any point on the surface of the bubble, the tangential velocity, \( V_t \), must be computed at each node in addition to the normal velocity, \( \delta \Phi / \delta n \). This is also done using a local surface fit to the velocity potential \( \Phi = g(x,y,z) \). Taking the gradient of this function at the considered node and eliminating any normal component of velocity appearing in this gradient gives a good approximation for the tangential velocity:

\[
V_t = \hat{n} \times (\bar{\nabla} \Phi \times \hat{n}) \tag{8}
\]

With the problem initialized and the velocity potential known over the surface of the bubble, an updated value of \( \delta \Phi / \delta n \) can be obtained by performing the integrations outlined above and solving the corresponding matrix equation. The unsteady Bernoulli equation can then be used to solve for \( D\Phi /Dt \), the total material derivative of \( \Phi \).

\[
D\Phi = \frac{D\Phi}{Dt} = \frac{\delta \Phi}{\delta t} + \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial t} = \frac{P_a - P_v}{\rho} - \rho g z + \frac{1}{2} |\bar{\nabla} \Phi |^2 + \frac{1}{2} |\bar{\nabla} \Phi |^2 \tag{9}
\]

\( D\Phi /Dt \) provides the time variations of \( \Phi \) at any node which is followed during its motion with the fluid. Using an appropriate timestep all values of \( \Phi \) on the bubble surface can be updated using \( \Phi \) at the preceding time step and \( D\Phi /Dt \). Here the timestep is based on the ratio between the smaller size of all panel sides and the highest node velocity. This choice has the great advantage of constantly adapting the timestep, by refining it at the end of the collapse, and increasing it during the slow
differences with these studies of the order of 0.1 percent on the bubble period. Finally, comparison with actual test results of the complex three-dimensional case presented here, Figure 4, shows strikingly similar complex bubble shapes.

Figure 1 shows the results of an axisymmetric case of bubble growth and collapse computed using the above described algorithm. It shows some general 3-D shapes and bubble profiles depicting the growth and collapse in a gravity field near an infinite horizontal plate above the bubble at a standoff ratio of \( L/R_{max} = 1.50 \), where \( L \) is the perpendicular distance from the initial bubble center to the plate. All coordinates have been normalized by \( R_{max} = 17 \text{cm} \). The figure shows how the bubble grows nearly spherically, then during the collapse phase flattens on the side which is opposite to the plate and forms a reentrant jet in a direction perpendicular to the plate. Note that for this case, the plate is located directly above the bubble, so both the buoyancy effects and the effects of the nearby boundary are in the same direction, vertically upward. The total period of the bubble, scaled by the Rayleigh time, \( R_{max} \sqrt{\Delta P/\rho g} \), was about 2.084. \( \Delta P \) is the difference between the ambient pressure and the minimum bubble pressure. This agrees very well with the bubble period given by Blake, et al. [9] who report a value of approximately 2.097 for a similar case computed in the absence of gravity with their axisymmetric scheme and a larger number of panels. The maximum velocity attained by the jet in this simulation was 11.1 \( \sqrt{\Delta P/\rho g} \). This too, compares very well with the results of other studies.

Figures 2 and 3 show the three-dimensional bubble growth and collapse when the infinite plate was moved to a vertical position adjacent to the bubble. The two cases were run when the plate was at standoff ratios of \( L/R_{max} = 1.50 \) and 1.00. With gravity acting vertically downward, the two competing forces (gravity, presence of plate) are perpendicular, rather than parallel to each other. The same initial discretization is used. Both of these figures show the formation of the reentrant jet moving at an angle upward toward the plate. As expected, the two competing forces not acting along the same line create a jet in a resulting direction depending on the proximity of the plate. In Figure 3, where the bubble is initially closer to the plate, the reentrant jet is more nearly perpendicular to the plate. In Figure 2, the angle of the reentrant jet is more acute and the jet penetrates the bubble and touches the other side at a point much closer to the top of the bubble than for the bubble in Figure 3. Also, one notices that during the collapse, once the jet has touched the other side of the bubble, the bubble retains a larger volume for the case where the plate is closer. Bubble periods are noticeably changed by the wall configuration relative to gravity. Placing the boundary vertically rather than horizontally results in an increased normalized period from 2.084 to 2.101. Moving the bubble closer to the wall lengthens the period still more. This is consistent with the known lengthening effect of the boundary and the reduction in period caused by the buoyancy force. A systematic series of runs for different plate angles was made and results on jet speed and angle cannot be shown here for lack of space available, but are summarized in reference [7].

Figure 4 shows the interaction between an unsteady bubble and an axisymmetric finite-size body. The body is cylindrical with two hemispherical end caps. The center of the submerged body was placed at the same depth as the initial center of the bubble and at a distance of 1.0 maximum bubble radii. Gravity and body attraction were acting perpendicularly to each other. The parameters for this case were chosen to match the experimental case shown in Figure 5 in Snay, Goerter and Price [10]. The computed bubble shapes closely resemble those actually recorded in the small scale underwater explosion test. A portion of the bubble is seen to adhere to the nearby body while the remainder behaves as if only gravity is the influencing factor. The one major discrepancy between the numerical and experimental results is the period of the growth and collapse of the bubble. The measured period is about 12 percent longer than the computed bubble period. This difference is mainly due to the fact that the experiments were conducted in a finite size cylindrical tank (diameter three times the maximum bubble diameter). The presence of the tank wall was shown by the numerical simulations to have a lengthening effect of the same order of magnitude on an otherwise isolated bubble.

The simulation of the growth and collapse of a bubble in a rotating flow field is illustrated in Figures 5 and 6. While implementing the nonuniform flow field method described above in the three-dimensional code, we have been testing the method using an existing axisymmetric code developed for bubble collapse near solid boundaries by Steve Wilkenson [7]. The code was modified to include the nonuniform (but here axisymmetric) flow and pressure fields of a Rankine vortex. The bubble is initially located at the vortex axis and, as for the cases described earlier, its growth is triggered by a prescribed initial internal gas pressure, \( P_g \), that is much larger than the ambient pressure.

The bubble behavior strongly depends on the values of \( \Omega \) and the core radius \( a_c \), since as can be seen from Equation (22), these control the pressure at the vortex axis and the amount by which it differs from the pressure distribution along the periphery of the bubble. At fixed values of \( \Omega \), Figures 5 and 6 show the strong influence of the core radius (or of the profile pressure versus radial distance from the vortex axis). Both figures show the bubble contour at equal times during both growth and collapse. The lower part of the figure shows the discretization points and can therefore be used to determine path lines of bubble surface particles.

As can be seen in Figure 5, during the initial phase of the bubble growth, radial velocities are large enough to overcome the swirl and the bubble grows almost spherically. Later on, the bubble shape starts to adapt to the pressure field and the bubble elongates along the axis of rotation. This elongation continues during the collapse phase while a constriction similar to a reentrant jet appears in the middle section of the bubble (plane of symmetry in this case where gravity is not included). The bubble then separates into two tear-shaped bubbles. It is conjectured that this splitting of the bubble into two is responsible for noise generation in vortex cavitation. This behavior is very similar to that observed experimentally for bubble growth and collapse between two plates [11] and obtained numerically using our 3-D program (see figure 7).

Keeping \( \Omega \) constant while reducing the core size \( a_c \) has the effect of steepening the radial pressure gradient along the bubble surface and increasing the rotation speed inside the viscous core (see Figure 8). This has the effect of increasing...
the deviation of the bubble shape from a sphere, and increasing the centrifugal force on the free-surface fluid particles closer to the vortex axis. As an undesirable consequence, near the axis the discretization of the bubble shape becomes poorer and poorer as time goes on and \( r \) decreases ( \( \Omega \) kept constant). It appears that the bubble tends to split into three bubbles but this cannot be confirmed until a scheme for sequential regridding of the bubble surface is implemented.

The above examples illustrate the capabilities of the developed method. The model is presently being extended to more general nonuniform flow fields, to bubble interactions and in a more accurate description of the latest phase of the bubble collapse when the reentrant jet approaches the opposite side of the bubble. Numerical instabilities occur in that last phase and can be reduced by smoothing techniques. Similarly, the pressures generated on nearby boundaries become unsteady and need to be more accurately determined.

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REFERENCES


FIGURE 2 - THREE-DIMENSIONAL BUBBLE GROWTH AND COLLAPSE.
L/T witness = 1.00, \( \frac{\rho_{\text{gas}}}{\rho_{\text{water}}} = 37, \frac{g_{\text{gas}}}{g_{\text{water}}} = 1300 \).
Vertical Wall, \( \text{Vertical Wall Time} = 0.0146 \text{sec} \).

FIGURE 3 - THREE-DIMENSIONAL BUBBLE GROWTH AND COLLAPSE.
L/T witness = 1.00, \( \frac{\rho_{\text{gas}}}{\rho_{\text{water}}} = 37, \frac{g_{\text{gas}}}{g_{\text{water}}} = 1300 \).
Vertical Wall, \( \text{Vertical Wall Time} = 0.0146 \text{sec} \).

FIGURE 4 - BUBBLE COLLAPSE NEAR ASYMMETRIC BODY
\( \text{A Frequency Discretization} \)
Figure 5. Bubble growth and collapse at the center of a vortex line. Large core radius.

Figure 6. Bubble growth and collapse at the center of a vortex line. Small core radius.

Figure 7. Bubble collapse between two parallel plates.

Figure 8. Schematics of velocity and pressure profiles versus the radius distance from the vortex axis. Viscous core radii of 1 and 2.