COLLECTIVE BUBBLE GROWTH IN A SUPERHEATED LIQUID FOLLOWING A SUDDEN DEPRESSURIZATION

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ABSTRACT

The presence and behavior of vaporous cavities are of major importance in many modern industrial applications where heat transfer boiling or cavitation are involved. Sudden depressurization of a superheated fluid is an example where bubble growth rate controls the heat transfer, and the transients involved. Most existing studies are concerned with individual spherical bubble growth as influenced by inertial and thermodynamic effects. Therefore existing computer modeling, and prediction codes neglect possible interactions and collective phenomena. This paper addresses this collective behavior using a singular perturbation approach. The method of matched asymptotic expansions is used to describe the bubble growth taking into account its interaction with a finite number of surrounding bubbles. A computer program is developed and the influence of the various parameters is studied numerically for a symmetrical equal size bubble configuration. A significant influence of interactions on bubble growth and heat transfer is observed. The growth rate of the bubble is reduced when it is in the presence of other bubbles, and the temperature drop at its wall is smaller than that for an isolated bubble, therefore the heat loss into bubble growth is smaller than expected. These effects increase with the number of interacting bubbles. A significant improvement of existing codes on transients in two-phase flow systems, could be obtained if these results are taken into account.
RESUME

La présence et le comportement de bulles de vapeur est d'une importance majeure dans beaucoup d'applications industrielles où le liquide est soumis à l'ébullition ou à la cavitation. La dépressurisation rapide d'un liquide surchauffé est un exemple où le grossissement des bulles contrôle les échanges thermiques, ainsi que les phénomènes transitoires en cours. La plupart des études existantes considère le grossissement de bulles sphériques contrôlé par les effets thermiques ou les effets d'inertie. Ainsi, les modèles numériques actuels négligent toute interaction ou tous effets collectifs possibles. Cet article étudie ces interactions utilisant une méthode de perturbations singulières. La méthode des développements asymptotiques raccordés est utilisée pour décrire le grossissement d'une bulle compte tenu de son interaction avec un nombre fini de bulles qui l'entourent. Un programme numérique est développé et l'influence des divers paramètres est étudiée numériquement pour une configuration symétrique de bulles de tailles identiques. Une influence significative des interactions sur le grossissement des bulles et sur les échanges thermiques est mise en évidence. Le taux de grossissement d'une bulle est réduit lorsqu'elle est en présence d'un nuage de bulles et la chute de température sur sa paroi est moindre que celui obtenu lorsqu'elle est isolée. En conséquence, la perte de chaleur en grossissement de bulles est moindre que prévu. Ces effets augmentent avec le nombre de bulles interagissantes. Si ces résultats sont pris en compte une importante amélioration des codes de calcul existants sur les transitoires dans les écoulements diphasiques est à prévoir.
INTRODUCTION

The presence and behavior of vaporous cavities are of great importance in many modern industrial applications where heat transfer, boiling or cavitation are involved. For instance, the rate of heat transfer in nucleate boiling depends essentially on the ability of the heat transfer surface to nucleate and support the growth of vapor bubbles. The conduction of heat in the liquid is greatly affected by the absorption and release of latent heat during the phase transition at the bubble-liquid interfaces. Wave propagation in the medium is affected significantly by bubble behavior and volume changes. Consequently, the study of the bubble dynamics and of the two-phase medium constituted by the host liquid and bubbles of its own vapor is fundamental in the design, analysis, and application of various engineering systems.

In cavitation inception (in the general sense: explosive growth of cavities) a key role is played by the initial presence of nuclei (minute cavities containing noncondensable gas) in the liquid. In hydrodynamic applications as with propellers, pumps, channels, pipes, submerged jets, etc., cavitation occurs when the liquid experiences significant pressure drops due to local high flow velocities. In this case the growth of the nuclei to macroscopic bubbles is mainly controlled by inertia. Heat transfer during the phase change at the bubble wall is negligible because of the low value of the vapor density at the ambient temperature, and does not affect the bubble-wall motion. The phase change is then called "cavitation" and the liquid is described as "cold". Conversely in applications where heat exchange, rather than the liquid inertia, controls the vapor-bubble dynamics, the phase change is called "boiling" [1-3] and the liquid is termed "hot".
Many modern processes such as those found in power plants, nuclear engineering, aerospace engineering, and the petrochemical industry deal with various fluids in conditions where they cannot be termed "cold", and where both thermodynamic effects and inertia contribute to controlling the bubble behavior. Examples of such fluids are hydrocarbons, liquid metals (sodium in particular), cryogenic fluids (such as liquid hydrogen, oxygen, and nitrogen), and demineralized hot water at temperatures as high as 300°C. Heat transfer boiling or cavitation appears with these liquids in such applications as high speed flows of sodium-cooled fast-breeder reactors in nuclear power engineering, circulation of cryogenic liquid in pumps in aerospace engineering, and flow of hot water in nozzles and tubes in steam power plants. Accidents such as loss of vacuum insulation in cryogenic storage tanks and loss of coolant in nuclear power plants, are sources of boiling nucleation and evidently of major safety concern [4]. Research studies in that area are very important and greatly needed.

This study addresses the behavior of a cloud of bubbles in a superheated fluid following a sudden depressurization. The problem of the growth of an isolated spherical bubble in an unbounded fluid has been extensively studied both for cavitation problems (see reviews in [1,2]) and for heat transfer boiling problems [2-10]. However, nonspherical bubble dynamics as well as the interaction between bubbles have been given much less attention. This is not due to the lack of recognition of the problem but to the complexity of the nonspherical free boundary value problem. Even though it is recognized that bubbles in a boiling liquid are seldom spherical and isolated, practically no studies have been done on the subject except for the general and very interesting formulation of Hsieh [11]. On the other hand, many experimental and theoretical investigations exist for nonspherical cavitation bubble collapse [1, 12] and a few
studies have been published on the collapse of a multibubble system or a bubble cloud [13, 14, 20]. Accounting for the collective effects in heat transfer boiling or in cavitation of superheated liquids is of major importance for the prediction of two-phase flows.

Rayleigh [16] in 1917 did the first analysis and Plesset [17] wrote the most comprehensive equation for inertia-controlled bubble growth and collapse. Later, Plesset and Zwick [5] and Forster and Zuber [6] derived an approximate analytical solution of vapor-bubble growth in a superheated liquid. In this approach a thin thermal boundary layer surrounding the growing bubble is introduced. Liquid vaporization at the bubble wall draws heat from the surrounding liquid whose temperature drops from its ambient value, $T_\infty$, to the value at the bubble wall in a distance corresponding to the thermal boundary thickness. This thickness is small compared to the bubble radius when the superheat $(T_\infty - T_b)$ is sufficiently large compared to a characteristic temperature of the fluid. $T_b$ is the equilibrium vapor temperature, (boiling temperature), of the liquid at the ambient pressure

$$P_\infty = p_v(T_b)$$  \hspace{1cm} (1)

In this case the bubble surface temperature, and thus the vapor pressure is obtained by an integral "memory" equation depending on the bubble growth history [5]. Recently, Dalle Donne and Ferranti [8] solved numerically the coupled momentum (Rayleigh-Plesset equation) and energy equation. Comparisons between the results of Plesset and Zwick based on a thermal boundary layer approximation [18], and the "exact" solutions of Dalle Donne and Ferranti [8] show very good agreement for Jacob numbers and superheats as low as 3 and 14 degrees respectively [4, 10]. The Jacob number is defined as:
\[ J = \frac{K (T_\infty - T_b)}{D L \rho_v(T_b)} \]  

where \( K \) is the liquid thermal conductivity, \( L \) the latent heat, \( \rho_v(T_b) \) is the equilibrium vapor density at the boiling temperature and \( D \) is the thermal diffusivity of the liquid. \((T_\infty - T_b)\) is the amount of superheat.

In previous work, we investigated analytically and numerically the inertia-controlled collapse of a bubble-cloud due to an increase of the ambient pressure [14, 15]. Two complementary approaches were considered: a singular perturbation theory and a continuum medium theory. The singular perturbation theory analyses the behavior of a finite number of bubbles interacting together during the collapse, and the acquired knowledge on their deformation and motion is being used for a two-phase continuum medium description [19].

In the work described in this paper, the singular perturbation approach for the study of the dynamics of a multibubble system is extended to the study of the growth of a bubble cloud in a superheated fluid where thermodynamic effects cannot be neglected. Both the more general approach and the boundary layer approximation are studied analytically, while numerical computations are conducted only in the second case.

ANALYTICAL MODEL

A. Formulation of the Problem

As a first step of the study of the general problem of a bubble cloud in a flow field and near solid boundaries, let us consider a cloud of bubbles in an unbounded medium of uniform pressure, \( P_\infty \), and temperature, \( T_\infty \). This corresponds to the
case where the size of the cloud is small compared to the flow field characteristic scales; $P_\infty$ and $T_\infty$ are then the local values of the pressure and the temperature in the flow field in absence of the cloud. A further simplification consists in neglecting the liquid viscosity and compressibility, and in assuming that the flow is irrotational. These assumptions are commonly accepted and are justified in cavitation and boiling heat transfer studies except in the last phases of the bubble collapse (see reviews in [1,2]).

The bubble cloud behavior is sought when the ambient pressure, $P_\infty(t)$, is time dependent. The liquid temperature and the ambient pressure are such that the liquid is superheated at least during a period of the pressure variations, i.e., the ambient temperature, $T_\infty$, is greater than the boiling temperature at these values of the ambient pressure, $T_b(P_\infty)$ (see equation (1)).

In order to determine the flow field in the bubble liquid medium and to obtain the motion and deformation of any bubble in the cloud one has to solve the Laplace equation,

$$\Delta \phi = 0 ,$$

subjected to kinematic and dynamical conditions on the bubbles' surfaces.

$$\nabla \phi \cdot n^i \bigg|_{r=R^i(\theta, \phi, t)} = \left[ \dot{R}^i e_{\tau} + \ddot{b} e_{z} \right] n^i$$

$$\rho [\dot{\phi} - \dot{b} e_{z} + 1/2 \mid \nabla \phi \mid^2 ] \bigg|_{r=R^i(\theta, \phi, t)} = P_\infty(t) - p^i_v + 2 \gamma C^i(\theta, \phi, t)$$

where $C^i$ and $n^i$ are respectively the local curvature of the
surface of bubble \( \mathbf{B} \) and its unit vector at the point \( M(r, \theta, \phi) \). The equation of the bubble surface in a coordinate system moving with velocity \( \mathbf{b} \) in the direction \( \mathbf{e}_z \), is \( r = R(\theta, \phi, t) \). \( \gamma \) is the surface tension and dots denote time differentiation. \( \phi \) and the operator \( \nabla \) are expressed in the moving coordinates system. We limit the study here to vapor filled cavities and neglect the presence of noncondensables. Due to the low value of the vapor density, \( \rho_v \), the pressure of the vapor inside the bubble can be assumed to be uniform as long as the spherical symmetry is preserved. In this case, \( P_v \) is equal to the value of the equilibrium vapor pressure at the bubble wall temperature. When the bubble deviates moderately from a sphere we will assume that both the temperature along the bubble wall and the value of the vapor pressure vary accordingly. Under this assumption, the pressure, \( P_v \), may be uniform far from the bubble surface but accomodates itself to the temperature controlled value in the vicinity of the interface. No more details on the way this happens inside the bubble are needed here since the flow field of the vapor is of no relevance as long as the velocities are subsonic.

Since the variation with temperature of the liquid density \( \rho \) and the surface tension, \( \gamma \), are small when compared with the variations of \( P_v \), the value of the equilibrium vapor pressure \( P_v(T_R) \) constitutes the main coupling between the dynamic and the heat problems. Solving the heat problem is needed to determine the temperature at the bubble wall, \( T_R(\theta, \phi, t) \), when temperatures deviate significantly from \( T_\infty \). The energy equation can be written:

\[
\dot{T} + \nabla \cdot \mathbf{v} = D \cdot \Delta T, \quad (6)
\]

where \( \mathbf{v} = \nabla \phi \) is the fluid velocity given by equations (3-5), and \( D \) is the thermal diffusivity of the liquid. Equation (6) is subjected to a boundary condition on the bubble wall stating that the heat locally lost at any point of the interface is
used to vaporize an amount of liquid determined by the local bubble volume expansion rate. This can be written:

$$ \frac{\partial T}{\partial n} \bigg|_{r=R_i(\theta, \phi, t)} = \frac{\rho_v L}{K} \cdot i $$

Equations (3) to (7) form with the initial and at infinity conditions (known $T_\infty$ and $P_\infty(t)$), a complete set of equations which must be solved to determine the flow and temperature field.

B. Asymptotic Approach

The solution of the general problem as presented in the preceding paragraph is not presently conceivable even numerically since it would involve considerable effort and computing time. However, when the bubbles' characteristic radius, $r_b$, is small compared to the characteristic distance between two bubbles, $l_o$, an approximate solution can be sought. For such low void fraction cloud, we can assume that interbubble interactions are weak enough so that, to the first order of approximation, and in the absence of relative velocity with the surrounding fluid, each of the individual bubbles reacts to the local pressure variations spherically, as if isolated. Mutual bubble interactions, individual bubble motions and deformations are taken into account in the following orders of approximation. Thus to first order, the cluster appears as a distribution of flow sources and heat sinks. To higher orders, any individual bubble in the cluster would sense the effect of the others first as an additional uniform velocity potential (and thus as an ambient pressure correction) and as a temperature change, and then as a uniform incoming velocity and a temperature gradient. In even higher order approximations velocity and temperature gradients have to be accounted for.
The solution of the problem is sought in terms of matched asymptotic expansions in powers of \( \varepsilon \), the ratio between \( r_{b_0} \) and \( l_o \). Two regions of the fluid are defined for each individual bubble of the cloud. The "outer problem" is that considered when the reference length is chosen to be \( l_o \). This problem is concerned with the macrobehavior of the cloud, and the bubbles appear in it only as singularities. The "inner problem" is that considered when the lengths are normalized by \( r_{b_0} \). The solution of this problem applies to the microscale of the cloud, i.e., to the vicinity of an individual bubble of center \( B_i \). The presence of the other bubbles, all considered to be at infinity for the "inner problem" is caused, at each order of approximation, by the asymptotic behavior of the outer solution in the vicinity of \( B_i \). Thus, to each order of approximation the "inner problem" reduces to the study of an isolated bubble with conditions imposed at infinity determined at the preceding order. By application of the matching principle these conditions are obtained as the expansions of the preceding order "outer solution" near the bubble singularity. The process is started by the first order approximation whose solution is known since all bubbles behave then as if isolated.

C. Nondimensionalizations

In order to make asymptotic expansions, an accurate choice of characteristic scale variables is fundamental. For the length scales the choice is immediate: \( r_{b_0} \) in the inner problem, \( l_o \) in the outer. \( r_{b_0} \) is chosen arbitrarily such that the inequality,

\[
\frac{r_{b_0}}{l_o} = \varepsilon \ll 1
\]  

is valid. When doing this we must keep in mind that the results of the computations will be valid only as long as the
radius of any bubble in the cloud does not greatly exceed $R_{b_0}$.

Concerning the time scale, the choice is simple once $R_{b_0}$ is known. In the case of a significant pressure drop, as will be the case for the problem of sudden depressurization in a Loss of Coolant Accident, the time scale is related to the pressure drop:

$$\tau_0 = \frac{r_{b_0}^{1/2} \rho^{1/2}}{\Delta P}.$$  \hspace{1cm} (9)

$\Delta P$ is the value of the sudden pressure drop, or could also be the order of magnitude of the pressure oscillation when $P_\infty(t)$ is a prescribed function of time. In the numerical example considered in this paper the pressure drops from its initial value, $P_{\infty_0}$, to a constant value $P_\infty$, in which case $\Delta P$ is defined as

$$\Delta P = P_{\infty_0} - P_\infty.$$ \hspace{1cm} (10)

As mentioned earlier, when the cloud is subjected to a sudden pressure drop the flow in the first approximation is that due to a distribution of dynamic sources and heat sinks. The characteristic strength of the dynamic sources is $q_0 = \frac{r_{b_0}^3}{\tau_0}$, and the resulting velocity potential is scaled with $q_0/\tau$. Thus, depending on whether one considers the "inner" or the "outer" problem, $\phi_0$ has the values:

$$\phi_0^{\text{in}} = \frac{r_{b_0}^2}{\tau_0}$$

$$\phi_0^{\text{out}} = \frac{r_{b_0}^3}{\ell_0 \tau_0}.$$ \hspace{1cm} (11)
Concerning the temperature scaling we know that the maximum temperature drop occurs near the bubble wall and that a lower bound for this temperature is the boiling temperature, $T_b$, of the liquid at the imposed ambient pressure, $P_\infty$. As a result, temperature departure from $T_\infty$ is scaled with the amount of superheat, $(T_\infty - T_b)$.

With these characteristic scales, nondimensional variables all of order unity are introduced and each of the unknowns is then expanded in power series of $\varepsilon$.

**SINGULAR PERTURBATION APPROACH**

A. **First Order of Approximations ($\varepsilon^0$)**

The determination of the flow field and the dynamics of any individual bubble, $B(i)$, are accessible once the boundary conditions at infinity in the corresponding "inner region" are known. When interactions are neglected, and in the absence of a slip velocity between the considered bubble and the surrounding fluid, the only boundary condition at infinity is the imposed ambient pressure variation $P_\infty(t)$. The "inner problem" is therefore spherically symmetrical and its solution is given by the well-known Rayleigh-Plesset equation [17]. This corresponds to the first order approximation of the problem considered here. To this order, ($\varepsilon^0$), the small parameter, $\varepsilon$, is equal to zero, and the normalized distance, $\varepsilon^{-1}$, between $B(i)$ and its neighbors with respect to the inner scale, goes to infinity. The bubble first-approximation nondimensional radius $a_i(t)$ is then given by the following Rayleigh-Plesset equation where the superscript (i) is omitted for convenience:

$$a_0 \ddot{a}_0 + \frac{3}{2} \dot{a}_0^2 = -\ddot{P}_\infty(t) + \Pi_0(t) + 2\dot{\omega}^{-1} \left( \frac{1}{R_0} - \frac{1}{a_0} \right) \quad \text{(12)}$$
\( R_0 \) is the initial bubble radius, and the nondimensional parameters are given by the relations:

\[
\tilde{P}_\infty(t) = \frac{P_\infty(t) - P_\infty(0)}{\Delta P},
\]

\[
W_e^{-1} = \frac{\dot{\gamma}(t)}{\frac{\Delta P}{r_{b_0}}},
\]

\[
\pi_o(t) = \frac{P_V(t) - P_V(0)}{\Delta P},
\]

where \( P_\infty(t) \) is the imposed ambient pressure, and \( \Delta P \) is the characteristic size of the pressure variations. \( \gamma(t) \) and \( P_V(t) \) are the surface tension coefficient and the vapor pressure respectively at the temperature of the bubble wall at time \( t \).

For a given \( P_\infty(t) \), equation (12) or (13) can be solved for the variations of the bubble radius, \( a_i(t) \). This allows the subsequent determination of the higher order approximations for the bubble radius. When the temperature at the surface of the bubble departs significantly from the ambient, and for liquids where the dependence of \( P_V \) and \( \gamma \) on temperature is important, it is necessary to couple equation (12) with the heat equation to obtain a solution.

After accounting for the spherical symmetry of the problem at this order the energy equation (6) reduces in first approximation to the following nondimensional equation:
The equation for the heat balance on the bubble-liquid interface can be formulated by stating that the heat locally lost at any point of the interface is used to vaporize an amount of liquid determined by the local bubble volume expansion rate. Since the problem is spherical at this order this condition, expressed by equation (7), becomes with nondimensional variables,

\[ \frac{\partial T_o}{\partial t} - \frac{a_o^2}{r^2} \frac{\partial T_o}{\partial r} = P_e \frac{1}{r^2} \left( \frac{\partial}{\partial r} \right) \left( r^2 \frac{\partial T_o}{\partial r} \right), \quad (14) \]

where the Peclet number, \( P_e \) is related to the thermal diffusivity, \( D \), the bubble characteristic size, \( r_b \) and the time scale \( \tau_o \) by the relation

\[ P_e = \frac{r_b^2}{D} \frac{\tau_o}{\Delta P} \frac{\rho_v^{1/2}}{D \rho^{1/2}}. \quad (15) \]

The equation for the heat balance on the bubble-liquid interface can be formulated by stating that the heat locally lost at any point of the interface is used to vaporize an amount of liquid determined by the local bubble volume expansion rate. Since the problem is spherical at this order this condition, expressed by equation (7), becomes with nondimensional variables,

\[ \frac{\partial T_o}{\partial t} = \left( \frac{\rho_v L}{K} \right) \cdot \left( \frac{r_b^2}{\tau_o T_{\text{ref}}} \right) \cdot \hat{\delta}_o = A \hat{\delta}_o. \quad (16) \]

where \( K \) and \( L \) are respectively the thermal conductivity and the latent heat of vaporization of the liquid, and \( \rho_v \) the density of its vapor at the bubble wall temperature. \( T_{\text{ref}} \) is the reference temperature which, as stated before, is chosen equal to the amount of superheat,

\[ T_{\text{ref}} = T_\infty - T_b. \quad (17) \]
B. Interactions

Due to bubble interactions, the local pressures and temperatures driving the growth of any bubble \( B(i) \) are, in the asymptotic theory presented here, a perturbation of the imposed far field pressure, \( P_{\infty}(t) \), and temperature, \( T_{\infty} \). Since these perturbations are due to the presence of the other bubbles in the flow field, the leading terms can be obtained directly once the first order behavior of all the bubbles in the cloud is determined. For instance, once equation (12), (14), and (15) are solved, the variations with time of the radius, \( \tilde{a}_j(t) \), of any cavity in the cloud is determined. This allows the determination of the intensity of any source \( \tilde{q}_j(t) \), by the relation

\[
\tilde{q}_j(t) = \tilde{a}_j^2 \tilde{a}_o
\]

which reproduces the flow due to the first order spherical motion of any bubble \( B(j) \). Consequently, the resultant "outer" potential flow is determined to this order by:

\[
\tilde{\Phi}_o(M, t) = \sum_{j=1}^{N} \frac{\tilde{q}_j(t)}{|MB_j|}
\]

where \( M \) is a field point, and \( B_j \) the center of the bubble \( B(j) \), also location of the source \( (i) \). The asymptotic expansions of \( \tilde{\Phi}_{\text{out}}(M, t) \), when the distance \( |MB_i| = \varepsilon \tilde{r}_i \), between \( M \) and a particular bubble \( B(j) \) goes to zero, contains additional terms to the leading source term, \( \tilde{q}_i/\tilde{r}_i \), corresponding to the order zero "inner" potential flow:

\[
\tilde{\Phi}_o = \frac{\tilde{q}_i(t)}{\tilde{r}_i}
\]
The other lower order terms express the interactions and are responsible for the flow and bubble shape corrections. For instance, by application of the matching principle (n - m rule [23]), the order $\varepsilon$ term will constitute the limit at infinity of the order $\varepsilon$ velocity potential in the "inner" problem, (i). This can be written as:

$$\lim_{r \to \infty} \phi_1 = \frac{x_0}{x_{ij}} q_j$$

(21)

where $x_0$ is the characteristic distance (used for scaling) between two bubbles and $x_{ij}$ is the actual initial distance between the two cavities' centers $B_i$ and $B_j$.

The first correction, $\phi_1$, of the undisturbed flow, $\phi_0$, has to satisfy the Laplace equation (3) as well as boundary conditions on the surface of the bubble $B_i$, (which are the contributions to order $\varepsilon$ of the expansions in powers of $\varepsilon$ of conditions (4) and (5) when nondimensionalized). This also applies to the first correction, $T_1$, of $T_0$ which satisfies the equations derived from (6) and (7).

As seen from condition (20) the dynamical problem remains spherical to this order, since the effect of the other bubbles does not introduce any disymmetries, and only changes the level of the potential velocity. Physically this means, that, since the problem is unsteady, the imposed "at infinity" pressure for the "inner" problem is modified by the time derivative of expression (21). As a result, the dynamical solution at order $\varepsilon$ is another source term which corrects the leading term $\phi_0$ given by (20). This solution can be written as

$$\phi_1 = \frac{q_i(t)}{r} + \sum_{j \neq i} \left( \frac{x_0}{x_{ij}} \right) q_j(t)$$

(22)

where $q_i(t)$ and $q_j(t)$ are functions of time. $q_o$ has been defined earlier in (20), and
The first spherical correction, \( \tilde{a}_1^i \), of the bubble radius is obtained by solving the following differential equation, where the superscript \( i \) and the tildes have been omitted:

\[
q_1 = a_o^2 \tilde{a}_1^i + 2a_o a_i \frac{\tilde{a}_1^i}{a_o} a_i^i \quad .
\]  

(23)

\( \pi_1(t) \) is a correction of \( \pi_0(t) \) defined in (13) and expresses the second approximation of the vapor pressure at the bubble wall. If the temperature is expanded like the other variables in powers of \( \varepsilon \), \( \pi_0(t) \) can be expressed as

\[
\pi_0(t) = \frac{[p_v(T_o(a_o,t)) - p_v(T_w)]}{\Delta P} ,
\]  

(25)

and \( \pi_1(t) \) as,

\[
\pi_1(t) = \frac{[p_v(T_1(a_o,t)) + a_1 \frac{\partial T}{\partial r} (a_o,t) \cdot \frac{dp_v}{dT} (T_o(a_o,t))]}{\Delta P} ,
\]  

(26)

The first correction of the temperature, \( T_1 \), is given by an equation similar to (14), which after omitting the superscripts (i) can be written as:

\[
\frac{\partial T_1}{\partial T} + \frac{\partial \phi_0}{\partial r} \frac{\partial T_1}{\partial r} + \frac{\partial \phi_1}{\partial r} \frac{\partial T_1}{\partial r} = \rho (-1) \frac{1}{\varepsilon r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T_1}{\partial r} \right) .
\]  

(27)

Similarly, the order \( \varepsilon \) heat balance equation at the bubble wall can be written as:

\[
\frac{\partial T_1}{\partial r} = A \tilde{a}_1 - \frac{\partial^2 T_1}{\partial r^2} a_1 .
\]  

(28)
Continuing the same procedure as in the preceding paragraph one can derive the equations for the flow field, the temperature field and the bubble motion. Considering the elementary solutions (spherical harmonics) of the Laplace equation, the boundary condition at infinity for any particular "inner" problem (i), can be shown to be up to order $\varepsilon^3$:

$$
\lim_{r \to \infty} \phi^i(M,t) = \sum_{i \neq j} \left\{ -\varepsilon \lambda^i_{ij} q^j_o - \varepsilon^2 \left( \lambda^2_{ij} q^j_o \cos \theta^i_{ij} + \lambda^i_{ij} q^j_o \right) + \varepsilon^3 \left[ \lambda^3_{ij} q^j_o \sigma^2 P_2(\cos \theta^i_{ij}) + \lambda^2_{ij} q^j_o \sigma^i_{ij} + \lambda^i_{ij} q^j_o \right] + \ldots \right\}, \quad (29)
$$

where,

$$
\lambda^i_{ij} = \left( \frac{\lambda^i_{ij}}{\lambda^i_{ij}} \right). \quad (30)
$$

The superscript $(j)$ denotes quantities corresponding to the other bubbles, $B(j)$. $\lambda^i_{ij}$ is the initial distance between the bubble centers $B_i$ and $B_j$. $\theta^i_{ij}$ is the angle $MB_iB_j$ and $r$ the distance $B_iM$, where $M$ is a field point in the fluid (see Figure 1). $P_n(\cos \theta)$ is the Legendre polynomial of order $n$ and argument $\cos \theta$. $q^j_o$ is the correction at order $\varepsilon^n$ of the strength, $q^j_o = a^j_o(a^i_o)^2$, of the source representing the first-approximation spherical oscillations of the bubble $B(j)$. Expressed in physical terms (velocities, pressures), the boundary condition (29) states that the first order correction, $O(\varepsilon)$, to the nonperturbed spherical behavior, $a^i_o(t)$, of the bubble $B(i)$ is a spherical modification of the collapse driving pressure. This introduces, as we have seen in the preceding section, a spherical correction $a^i_{ij}(t)$ of the radius variations $a^i_o(t)$. At the following order, $\varepsilon^2$, a second correction of the uniform pressure appears, and a uniform velocity field accounting for a slip velocity between the bubble and the surrounding fluid is to be added. This induces
a spherical correction, \( a^i_2(t) \), of \( a^i_0(t) \), and a nonspherical correction \( f^i_2(t) \cdot \cos \theta^i g \), where \( \theta^i g \) is an angle which can be compounded from all the \( \theta^i j \). Things become more complex at the order of expansion \( \varepsilon^3 \), where in addition to the uniform pressure and velocity corrections, \( a^i_3(t) \) and \( f^i_3(t) \cdot \cos \theta^i g \), a velocity gradient generated by the flow field associated with the motion of all the other bubbles, is to be accounted for, to generate a nonspherical correction, \( g^i_3(t) \cdot [3 \cos^2 \theta^i g - 1]/2 \).

Thus, the equation of the surface of the bubble \( B(i) \) expanded in Legendre polynomials can be written in the form

\[
R^i(\theta^i g, t) = a^i_0(t) + \varepsilon a^i_1(t) + \varepsilon^2 [a^i_2(t) + f^i_2(t) \cdot \cos \theta^i g] + \\
+ \varepsilon^3 [a^i_3(t) + f^i_3(t) \cdot \cos \theta^i g + g^i_3(t) \cdot P_2(\cos \theta^i g)] + ... ,
\]

provided that the initial bubble shape is spherical. We see that, up to the order \( \varepsilon^3 \) the problem is axisymmetric, and the axis of symmetry is for every bubble in the direction \( B^i G \) of its motion towards the bubble cloud "center" (see Figure 1). Similarly, the temperature is expanded as follows:

\[
T(r, \theta, t) = T_0(r, t) + \varepsilon T_1(r, t) + \varepsilon^2 [T_20(r, t) + T_21(r, t) \cdot \cos \theta] + \\
+ \varepsilon^3 [T_30(r, t) + T_31(r, t) \cdot \cos \theta + T_32(r, t) \cdot P_2(\cos \theta)] + O(\varepsilon^3).
\]

Similar equations to (12), (14), (24), and (27) are obtained for the higher order terms of \( R(\theta, t) \) and \( T(\theta, t) \), and are not given here for reason of brevity. For details please consult a more detailed report prepared for the National Science Foundation [22].
NUMERICAL RESOLUTION

The system of equations (12), (14), (24), (27), (40) and following, up to order \( \varepsilon^3 \), constitute a set of 14 equations, for the 14 unknown components of \( R(\theta,t) \), and \( T(\theta,t) \) (expansions (31) and (32)). By solving this system one determines completely the flow and temperature fields as well as the bubble motion. A numerical solution of these equations is feasible and could be performed using the same procedure as Dalle Donne and Ferranti [8]. Their study dealt with a single bubble growth and thus solved only equations (12) and (14). Here the same approach would have to be performed for all seven components of the bubble radius (up to \( \varepsilon^3 \)) so the computation time would be multiplied by at least a factor of seven. However, no additional problems are expected, since all dynamical equations on one hand and heat equations on the other are basically of the same type.

If the assumption of the existence of a thin thermal boundary layer on the bubble wall is considered, an approximate solution can be obtained more easily. This approximation is valid as long as the Jacob number, \( J \), is larger than one. Comparison between numerical computations for a single bubble obtained using this approximation and those obtained by solving the general equations gave very close agreement for \( J \geq 3 \), [2,4,10].

When the boundary layer approximation is made the system of heat equations, presented above, simplifies considerably. Indeed, in that case temperature departs from \( T_\infty \) only in the liquid region close to the bubble-liquid interface, and the values of \( r \) which are of interest are close to \( R(\theta,t) \). The Lagrange variable, \( y \), is then small compared to \( a_0^3 \), and we can write
where $\tilde{y}$ and $a_0$ are of order 1, while $\xi \ll 1$. The problem contains then two small parameters $\epsilon$ and $\xi$. An asymptotic solution uniformly valid when both $\epsilon$ and $\xi$ go to zero can be obtained when a relationship between the two parameters is defined through the use of the principle of least degeneracy [24].

Considering the heat equation, one can determine the needed relation between $\epsilon$ and $\xi$ to conserve the maximum number of terms in the leading orders of approximation. The Peclet number has to be large enough to satisfy

$$P_e^{-1} \xi^2 = O(1) \quad ,$$

and, we need

$$\xi = O(\epsilon) \quad .$$

In that case the expansions become straightforward.

The equations obtained at the first order expansion in both parameters (orders $\epsilon^0$ and $\xi^0$) are those for the case of an isolated bubble. These equations are the Rayleigh-Plesset equation, (12), and the following heat equation

$$\dot{T}_o - P_e^{-1} \frac{\partial}{\partial y} (a_0^4 \frac{\partial T_o}{\partial y}) = 0 \quad .$$

A solution is readily available in that case and was derived by Plesset and Zwick [5] and Forster and Zuber [6], using Laplace transform methods. The nondimensional temperature at the bubble wall is given by
\[ T_0(q_0,t) = T_\infty - \left( \frac{b_0^2 D}{2\pi t_0} \right)^{1/2} \int_0^t \frac{a_0^2(x)}{[\int_t a_0'(y)dy]^{1/2}} \frac{\partial T_0(q_0,x)}{\partial r} \, dx , \]  

(37)

where \( \partial T_0/\partial r \) is determined at the bubble wall by the boundary condition (16). One finally obtains the equation

\[ T_0(q_0,t) = T_\infty - \left( \frac{D}{\pi t_0} \right)^{1/2} \frac{b_0}{K(T_\infty - T_b)} \int_0^t L(x)\rho_v(x) \frac{a_0^2(x)}{\int_t a_0'(y)dy} \frac{\partial T_0(q_0,x)}{\partial r} \, dx , \]  

(38)

where, to be consistent with the assumptions made in deriving this solution [2,5] \( D \) and \( K \) are constant and evaluated at \( T_b \), while \( L \) and \( \rho_v \) are functions of time. Plesset and Zwick [5] also gave the solution of the problem when equation (36) contains a right hand side which is a known function of time (heat source term). Using a matched asymptotic procedure they also computed the following order of approximation, \( 0(\xi) \). These solutions correspond to the following orders equations obtained here in this problem.

The numerical procedure is most simplified now that the analytical expressions for the temperature at the bubble wall are known. The finite element method which would have been used in the general case is here replaced by a numerical computation of the integral equation (38). An iteration procedure is required to insure that the computed value of \( T_0(q_0,t) \) does not differ significantly from the value presumed in the computation of the integrand.
NUMERICAL ILLUSTRATION OF THE METHOD

A. Particular Cases Studied

In order to illustrate the method presented above we consider a simple geometry for the cloud. The bubbles are distributed in a symmetrical configuration and are initially of equal size. With this configuration all bubbles behave identically. As a result, the computation time is significantly reduced. All the bubbles have the same radius history, and no repeated computations for each bubble are needed. The computation is further simplified by the fact that all summations as those in equations (24) and (29) reduce to multiplications of the characteristics of a single bubble by three constants which depend only on the initial geometrical configuration:

\[ c_1 = \sum_{i \neq j} \lambda_{ij}, \quad (39) \]

\[ c_2 = \sum_{i \neq j} (\lambda_{ij})^2, \quad (40) \]

\[ c_3 = \sum_{i \neq j} (\lambda_{ij})^3, \quad (41) \]

An additional simplification of the problem can be introduced if one notices that during the bubble growth the departure from the spherical shape happens very late in the bubble history and only when the asymptotic approach starts losing its validity. Based on this observation we can neglect the non-spherical part of the bubble surface equation when solving the heat transfer problem. We can then consider that equation (38), relating the deviation of the bubble wall temperature from the ambient temperature, with the spherical bubble radius history, is applicable here to the spherical part of the bubble radius, i.e., \( \sum \varepsilon^n a_n(t) \).
With this simplification, at any time step all dynamical equations are solved using the value of the vapor pressure corresponding to the liquid temperature at the bubble wall. This temperature is computed at the preceding time step using equation (38). The nonspherical part of the bubble shape is not disregarded and is computed neglecting any variation of the liquid temperature along the bubble surface. This is valid as long as the bubble deformation is negligible. Since we restrict this study to that case, the validity of the results is monitored by checking the relative value of the computed nonspherical to the spherical components of the bubble surface equation. The computation is stopped when an imposed limit is exceeded.

B. Results and Interpretation

A series of numerical cases was studied using a VAX11/750 computer. Figures 2 to 11 illustrate the kind of results obtained on the influence of bubble interactions on the growth of a bubble in a superheated liquid. Figure 2, shows clearly this influence on the bubble growth. Since the bubble does not stay spherical, the value of $R(\theta)$ represented in this figure corresponds to the point the bubble closest to the cloud center. This value of $R$ is called the "lower-minor radius" in Figures 7 to 10 which examine the bubble deformation. Figure 2 using nondimensional variables shows first the classical results of asymptotic growth in $t$ for the inertia-controlled expansion and in $t^a$ for the heat-controlled expansion. If there was no pressure drop $a$ would be 1/2. However, as seen here, $a$ is much closer to 1, as obtained by earlier studies on single bubbles [9].

The most important result obtained here is that bubble growth is inhibited by bubble interactions. We observe very
clearly that the bubble size decreases with the number of interacting bubbles. This decrease exceeds 20 percent for a 5 bubbles system for nondimensional times larger than 10, or one millisecond after the start of the growth (Figure 5).

Figures 3 and 4 being in logarithmic scales show clearly the various stages of bubble growth. The time delay for significant growth (10^-8 s, in this case) is presently a subject of research [24]. In the case presented here, most of the inertia-controlled growth occurs at the end of this early stage. The growth rate increases linearly before attaining the heat-controlled stage where this rate becomes constant. Most of the temperature drop at the bubble wall occurs during this stage (Figure 5). In the last phase, the growth rate decreases and the bubble radius deviates from the power law curve as temperature at the bubble wall approaches the boiling temperature [8, 10]. In Figure 5, we can see how the growth rate is significantly reduced in this last stage when the number of bubbles increases.

Figure 6 shows the effect of bubble interactions on the liquid temperature at the bubble wall. The presence of other growing bubbles in the field is seen to reduce the heat transfer at the bubble wall and thus the temperature drop in the vicinity of the bubble. For a 5-bubbles system the magnitude of the temperature drop is more than 30 degrees a millisecond after the pressure drop. This result coupled with that on the bubble radius is important for any practical computation of heat transfer in a two-phase medium.

Figures 7 to 10 show the modification of the bubble shape during its growth for the various multibubble systems studied (2, 3, 5, and 12 bubbles). Represented are the variations of \( R(\theta) \) for three values of \( \theta \). Since here we are considering configurations where the bubble centers are distributed symmetrically on the surface of a sphere, the "minor radii" are on
the bubble's axis of symmetry which is normal to the surface of the sphere. The point on the bubble and on this axis closest to the center of the sphere defines the "lower minor radius" while the point farthest defines the "upper minor radius". The "major radius" is that obtained in the perpendicular direction to the "minor radii" and corresponds to a tangential direction to the sphere. As expected the side of the bubble facing the cloud center is seen to be "pushed away" from the cloud center and the bubble to elongate in a direction tangential to the sphere. Any point on its surface remains however, always inside the corresponding fictitious isolated bubble growing under the same conditions. The deformation decreases as the number of interacting bubbles increases.

The last figure considers the influence of the initial bubble size for given pressure conditions. With the assumption that the bubble is initially at equilibrium, the modification of the initial bubble size corresponds also to a change of the amount of superheat. Figure 11, shows the same effect with nondimensional variables and compares a 5-bubbles system with the isolated bubble case. One can notice that the inhibition effect due to bubble interactions is larger for smaller initial bubbles or larger amounts of superheat.

**CONCLUSION**

We have presented here a theory for the growth of a cloud of bubbles in a superheated liquid. To do so we have used a matched asymptotic expansion assuming a low void fraction (small ratio of bubble radius to interbubble distance). Numerical solutions were then obtained using a multi Runge-Kutta scheme to solve the dynamical equations and a thin thermal boundary integral solution for the energy equations. The results obtained show that a significant influence of bub-
ble interactions on bubble growth and heat transfer exists. The effects of this influence can be summarized as follows:

a. The growth rate of the bubbles is reduced,

b. The radius of any bubble at a given time is smaller than would be found for an isolated bubble,

c. The temperature drop at the bubble wall is smaller at any given time than would be found for an isolated bubble.

These effects increase with the number of interacting bubbles, as well as with the amount of superheat and pressure drop. These results obtained using small perturbations assumptions are expected to remain valid and become more significant when the void fraction becomes larger. Accounting for these deviations from the classical spherical results is important for increasing the accuracies of the existing transient two-phase flow codes [4, 21].
REFERENCES


21. IAHR 4th and 5th International Symposium on Water Column Separation, held successively at Cagliari, Italy, September 1979 and Obernach, Germany, September 1981.


FIGURE 1 - MULTIBUBBLE INTERACTION EQUIVALENCE CONCEPT
FIGURE 2 - INFLUENCE OF INTERACTIONS ON BUBBLE GROWTH IN A SUPERHEATED LIQUID

FIGURE 3 - BUBBLE GROWTH IN A SUPERHEATED LIQUID
FIGURE 4 - BUBBLE GROWTH RATE FOR A SYMMETRICAL BUBBLE SYSTEM

FIGURE 5 - INFLUENCE OF BUBBLE INTERACTIONS ON THE BUBBLE GROWTH RATE
Figure 6 - Influence of Interactions on the Temperature Variations at the Bubble Wall
FIGURE 7 - VARIATION WITH TIME OF BUBBLE SHAPE CHARACTERISTICS FOR A 2 BUBBLE SYSTEM

FIGURE 8 - VARIATION WITH TIME OF BUBBLE SHAPE CHARACTERISTICS FOR A 3 BUBBLE SYSTEM
FIGURE 9 - VARIATION WITH TIME OF BUBBLE SHAPE CHARACTERISTICS FOR A 5 BUBBLE SYSTEM

FIGURE 10 - VARIATION WITH TIME OF BUBBLE SHAPE CHARACTERISTICS FOR A 12 BUBBLE SYSTEM
FIGURE 11 - INFLUENCE OF INITIAL BUBBLE SIZE AND BUBBLE INTERACTIONS ON RADIUS HISTORY