INTRODUCTION

A remarkable change is observed in the structure of a submerged cavitating jet when the jet is very slightly excited periodically at frequencies \( \delta \), corresponding to Strouhal numbers \( (\delta = f d/V \), \( d \) jet diameter, \( V \) jet mean velocity) near 0.3 or one of its first integer multiples. The aspect of the cavitation field in the shear layer is modified from scraggly random quasi-linear vapor vortices to structured, well-organized ring-vortex cavities moving along the jet periphery. A better knowledge of the behavior of cavitating vortex rings in an infinite medium motivated our previous studies \( (1-3) \). This paper presents an extension of these studies, and considers the influence of a solid wall on the collapse of a toroidal bubble when submitted to a sudden pressure jump.

MODEL PRESENTATION

The initial geometry of the problem is the following: the ring section characteristic radius is \( R_0 \), its overall radius is \( A_0 \), and its distance from the solid wall is \( d_0 \) (Figure 1). The fluid is assumed to be inviscid and incompressible; and the bubble is animated by a vertical motion of constant circulation. The bubble is initially at equilibrium and approaches the wall at its self-induced velocity. It contains noncondensable gases of partial pressure \( P_{\text{g}} \) and liquid vapor of partial pressure \( P_{\text{l}} \). The balance between the internal and the external pressures at the bubble wall is due to the surface tension, \( \gamma \). The dynamic response of the toroidal bubble to a sudden ambient pressure step of amplitude \( \Delta P \) is sought.

When the ring cross-section characteristic size, \( R_0 \) is of the same order as its overall radius \( A_0 \), the problem is not easily solvable analytically, and at static equilibrium the shape of the bubble section is not circular. However, when \( \epsilon = R_0/A_0 \ll 1 \), and when \( d_0 > R_0 \), an asymptotic theory can be successfully applied to solve the problem. In that case, a circular equilibrium bubble section is obtained in first approximation. Furthermore, the existence of two length scales, \( R_0 \) and \( A_0 \), allows the use of matched asymptotic expansions to derive an approximate solution of the problem.

In the "outer region" (macroscale \( A_0 \)), the torus appears as a distribution of singularities on a moving circle of radius \( A(t) \). The presence of the solid wall is taken into account by the use of the method of images: the wall is replaced by a symmetrical singularity distribution. In the "inner region" (microscale \( R_0 \)), the torus is reduced in first approximation to an isolated cylindrical bubble of variable section \( R(t) \). At higher orders of approximation, deviation from the circular symmetry and motion of the vortex ring appear and are computed. In what follows we will limit our study to the case where the initial distance to the wall, \( d_0 \), is of the same order as \( A_0 \). An analysis of other cases can be found in reference \( (3) \).

BUBBLE RING DYNAMICS

With the above assumptions, the flow is potential and we have to find the velocity potential, \( \phi \), which satisfies the Laplace equation, and the boundary conditions at infinity on the toroidal bubble wall and on the solid wall. In a matched asymptotic approach, the two boundary conditions at the bubble wall - namely the dynamic condition - apply only in the "inner problem," while the at-infinity condition and the solid wall boundary condition are to be verified only in the "outer problem." The two problems, consequently are linked with a matching condition.

In order to simplify the analytical formulation, we follow the bubble during its motion and we subject the frame of reference to move with the vortex. Therefore, the origin of coordinates in the inner problem is also the location of the vortex center. This translation velocity is noted \( V_c \), with \( x \) and \( z \) its \( x \)-axis and \( z \)-axis components.
a. First Order Approximation

In the "inner problem," the bubble surface is reduced to the cylinder \( r = R(t) \), and the kinematic condition on this surface can be written

\[
\sum_{\text{in}} \cdot \mathbf{n} = (R \mathbf{e}_\mathbf{r} + \mathbf{V}) \cdot \mathbf{n}
\]

(1)

where dots denote time derivatives and \( \mathbf{n} \) is the normal unit vector to the bubble surface.

Using the Bernoulli equation, one obtains the dynamic condition at the bubble liquid interface

\[
\rho (\mathbf{V}^2 + \mathbf{V} \cdot \mathbf{V}) + p = \rho [2 \mathbf{V} \cdot (\mathbf{V} + \mathbf{V})] + \rho \mathbf{V} \cdot \mathbf{V} = \rho \mathbf{V} \cdot \mathbf{V} - \rho \mathbf{V} \cdot \mathbf{V}
\]

(2)

where \( \mathbf{V}_m \) is the mean curvature of the bubble surface.

In the "outer problem" the collapsing vortex ring and its image in the wall appear in first approximation as a distribution of sources and vortices on the circles \( C \) and \( C' \), moving toward each other at the relative velocity \( 2 \mathbf{V}_t \). If the unknown near strength of the sources is \( q(t) \) and the unknown intensity of the vortices is \( \omega \), the first order of the "inner problem" velocity potential \( \phi^\circ \) is determined at the point \( M \) by

\[
4 \pi \phi^\circ_{\text{out}} (M,t) = q(t) [\int_C \frac{ds}{r^2} + \int_{C'} \frac{ds}{r^2}] + \omega [\int_S \frac{\mathbf{V} \cdot \mathbf{n}}{r^2} ds - \int_S \frac{\mathbf{V} \cdot \mathbf{n}}{r^2} \mathbf{n} \cdot ds].
\]

(3)

Here \( S \) is a surface limited by the circle \( C, P \) and \( Q \) are field points respectively on \( C \) and \( S \).

The expansion of \( \phi^\circ_{\text{out}} \) when the field point \( M \) approaches the origin \( 0 \) can be expressed with the inner variables, \( \theta = \rho R_0 \) and \( \mathbf{e} \), as follows:

\[
\exp \phi^\circ_{\text{out}} = q(t) \left[ \frac{\alpha}{2} - aK(\alpha) + \frac{1}{3} \alpha \left[ \frac{\alpha}{2} - aK(\alpha) \right] + \frac{1}{2} \alpha \left[ \frac{\alpha}{2} - aK(\alpha) \right] \right] + \frac{1}{3} \alpha \left[ \frac{\alpha}{2} - aK(\alpha) \right] + \frac{1}{2} \alpha \left[ \frac{\alpha}{2} - aK(\alpha) \right] \}
\]

(4)

where \( a = \frac{\alpha}{2} \) and \( K(\alpha) \) are the complete elliptic integrals of the first and second kind, and \( \alpha \) is defined by

\[
p = \frac{\alpha}{2} \quad \alpha = p(1 + p^2)^{-1/2}
\]

(5)

Through the matching condition between the "inner" and the "outer" problem, the leading terms of Expansion (4) are the leading terms of the expansion of \( \phi^\circ_{\text{out}} \) when \( T \) goes to infinity. Since \( u \) is not time dependent, one can see from (4) that up to the order \( \epsilon \log(B/c) \), the velocities are independent of \( \theta \), and the cylindrical symmetry is conserved. If \( u \) and \( q(t) \) are of the same order, then both the radial expansion velocity, \( R \), and the vortical velocity, \( R/2\pi \), appear before the translation velocity, \( \mathbf{V}_t \). This latter is associated with the terms in \( \cos \theta \) and \( \sin \theta \) in equation (4). Then the inner problem can be solved at the first order with the superposition of the potentials of a two-dimensional source and a vortex. Using the at-infinity condition derived from expansion (4) and the kinematic condition (2), one obtains

\[
q(t) = 2\pi \frac{R_0^2}{T} \frac{d^2}{dt^2} \left( \frac{\mathbf{e}_r}{R} \right), \quad \omega = \mathbf{e}_r
\]

(6)

where \( R_0^2 \) is the first order of the bubble equation, \( R(0,t) = R(0,t)/R_0 \), and \( T \) is the characteristic time of the collapse.

Since the velocity potential is usually determined except for an additive constant, expansion (4) determines this missing constant for the inner potential. Its value is \( \phi^\circ = q(t) \log(c/8)2\pi \). Therefore, it becomes possible to obtain the first order equation for \( R^2 \) with the assumption that the non-condensables in the bubble obey an ideal compression-expansion gas law of polytropic coefficient \( K \).

\[
\frac{d^2R^2}{dt^2} + \frac{1}{2R^2} \left( \frac{dR}{dt} \right)^2 - \frac{dR}{dt} = \frac{q(t)}{R^2} \left( \log \frac{c}{R_0} \right) + \frac{1}{2R^2}
\]

(7)

The parameters in (7) are defined with:

\[
a_0 = (1+\epsilon_0)^{-1/2}, \quad \phi^\circ_{\text{out}}(t) = \left[ \frac{P(t)}{P(0)} - \frac{P_0}{P(0)} \right] / \Delta P,
\]

\[
\frac{d_0}{d_0} = \frac{P_0}{P(0)}, \quad \frac{d_0}{d_0} = \frac{P_0}{P(0)}, \quad \frac{d_0}{d_0} = \frac{P_0}{P(0)}, \quad \frac{d_0}{d_0} = \frac{P_0}{P(0)}, \quad \frac{d_0}{d_0} = \frac{P_0}{P(0)}, \quad \frac{d_0}{d_0} = \frac{P_0}{P(0)}.
\]

(8)

One obtains the value of \( T \) by balancing the predominant terms between the left and right hand side of the dynamic equation (7),

\[
T = \frac{R_0}{\epsilon_0} \left[ T_0 \right] \left( \log(8/c) \right)^{1/2}
\]

(9)

This result is the same as for a torus collapsing in an infinite medium (1) since it assumes \( a_0 \epsilon_0 K(a_0) \) is of order 1. In a more general case where \( \epsilon_0 \) is small compared to \( A_0 \), the characteristic time \( T \) increases when \( \epsilon_0 \) decreases. Its expression can then be written, provided that \( d_0 < 1 \),

\[
T_0 = \frac{R_0}{\epsilon_0} \left[ T_0 \right] \left( \log(32/c_0) \right)^{1/2}
\]

(10)

B. Following Orders: Ring Deformation and Motion.

The complete calculations leading to the following statements are given in (3). We explain hereafter the main steps needed to obtain the torus section shape deformation and its motion. Further expansions of the at-infinity condition (4) and the boundary conditions (1,2) show that the radius and the velocity potential in the inner problem can be written in the form

\[
\phi^\circ = \phi^\circ + \epsilon \log(B/c) \phi^\circ + \epsilon^2 \phi^\circ + \epsilon^3 \phi^\circ
\]

(11)
Classical Fourier series expansions can be used to describe \( R \):
\[
R^2 = \sum_{n=0}^\infty \left( A_n^1 \cos n\theta + B_n^1 \sin n\theta \right) .
\]

One can then see from the equation's expansions that the first correction of the velocity potential, \( \phi' \), of order \( \log(8/\epsilon) \), has to account only for the torus motion, and not for any bubble volume change correction. Therefore, it can be written:
\[
\phi'_i = \sum_n \left( A_n^1 \cos n\theta + B_n^1 \sin n\theta \right) .
\]

At the following order of approximation, \( \epsilon \), corrections of the growth potential appear and reflect the three-dimensionality of the problem, and \( \phi'_i \) is expanded as:
\[
\begin{align*}
\phi'_i &= \left( R^2 \cos \theta + \omega T R \cos \theta \right) \log T^2/2 + \\
&\quad + \sum_n \left( A_n^2 \cos n\theta + B_n^2 \sin n\theta \right) .
\end{align*}
\]

When the expansions (11) to (14) are accounted for in the equations of the problem, relations between the various constants \( A_n^1, B_n^1 \), \( A_n^2, B_n^2 \) are found. We will concentrate here on the bubble shape and give the results for \( A_n^1 \) and \( B_n^1 \). One finds for instance \( A_n^1 = B_n^1 = 0 \) for \( n=1 \). The remaining constants \( A_n^2, B_n^2 \) combine with the components \( 2^i, X \) in a motionless frame of the displacement of the moving origin of coordinates. If we define:
\[
C^i = A_1^i + 2^i, \quad D^i = B_1^i + X^i,
\]
we can write differential equations for \( C^i \) and \( D^i \). Space limitation does not allow presentation of the equations here.

Unfortunately, the above-presented procedure does not allow for the distinction between ring motion and deformation. The equation of conservation of the circulation and the equation of dynamic equilibrium of the torus do not bring additional information. We will analyze in what follows various arbitrary divisions between the motion and deformation parts of \( C^i \) and \( D^i \).

RESULTSD AND DISCUSSION

The obtained equations were solved numerically with a Runge-Kutta procedure. We present here the results obtained for a sudden increase of the pressure at infinity. Figure 2 shows the influence of the proximity of the solid wall on the bubble collapse. We plotted only the solution of the first order, \( R(t) \). It is easily seen that the wall has, already at this order, a lengthening effect on the bubble dynamic response. This effect seems to be very well described by equation (10).

In order to analyze numerically the different possible solutions of the problem, namely relative importance of motion and deformation, we define two numerical parameters \( \lambda_x, \lambda_z \) of values between 0 and 1, and impose arbitrarily the following decomposition:
\[
A^i(t) = \lambda_x [c^i(t) - c^i(0)] + \lambda_z [d^i(t) - d^i(0)] ,
\]
\[
z^i(t) = (1-\lambda_x) [c^i(t) - c^i(0)] + c^i(0) ,
\]
\[
x^i(t) = (1-\lambda_x) [d^i(t) - d^i(0)] + d^i(0) .
\]

The above decomposition satisfies the initial condition \( A_{10} = B_{10} = 0 \) (no initial deformation velocity). Besides, the initial values of \( C^i \) and \( D^i \) are assumed to be given by the values of the initial components of the translation velocities \( 2^i \) and \( X^i \). Obviously, our choice is not unique and any other decomposition could have been chosen. We present hereafter the solutions obtained for three combinations of \( \lambda_x, \lambda_z \).

In Figure 3, \( \lambda_z = \lambda_x = 0 \), and no shape deformation is allowed. The bubble shape remains circular and the torus is moving toward the wall while stretching. This solution is that obtained if we assume that the vortex center, which is inside the bubble, follows the vortexial fluid motion at that point. We can see that in this case it is possible to follow the ring collapse for dimensional time up to 1.06. After that point instabilities in the higher order terms invalidate the solution. In Figure 4, \( \lambda_z = \lambda_x = 1 \), no frame acceleration is allowed and the vortex center, also origin of coordinates, keeps its initial translation velocity. Here, shape deformation is very important. A reentering microjet begins to appear and is directed towards the wall and the center of the jet. The computation stops when the bubble interface (tip of the microjet) touches the origin of coordinates. This limits the computation time to \( t = 0.925 \).

Figure 5 presents the intermediary case with \( \lambda_x = \lambda_z = 0.5 \). The bubble's center and the coordinate system are accelerating towards the wall and a reentering jet develops in the same direction as the one in the previous case, but its appearance is delayed.

The nonuniqueness of the solution stems from its approximate character. Theoretically, all the results presented above would become identical if the number of terms of the expansions are infinitely increased. These results are very similar to those obtained for a spherical bubble collapsing near a solid wall (4).

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REFERENCES

